Maths Learning Service: Revision ***Mathematics IA***

Differentiation

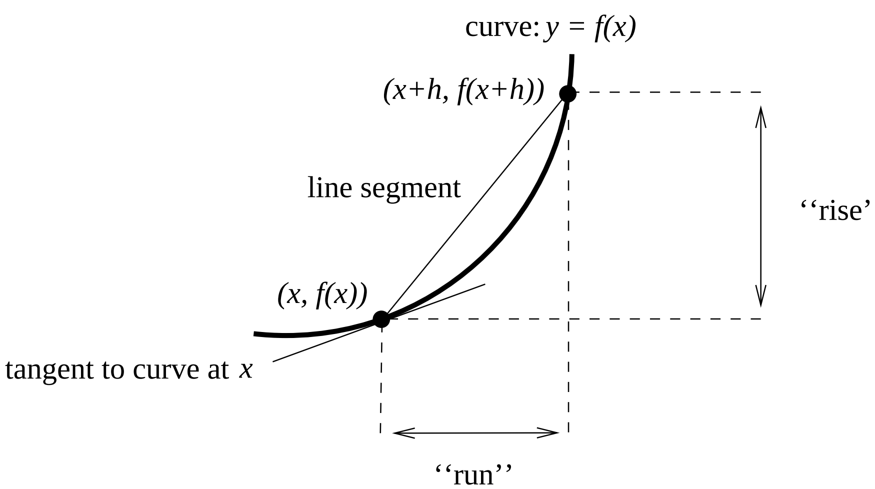
Mathematics IMA

The derivative

Consider a function $y = f(x)$. For some point x, we can find

- the slope of the tangent to the curve described by $f(x)$, or
- the instantaneous rate at which y is changing

by the following method: Find the slope of the line segment joining $(x, f(x))$ and a nearby point $(x + h, f(x + h))$ as shown below:



So,

$$\frac{\text{“rise”}}{\text{“run”}} = \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{x + h - x}$$

and, in the limit as $h \to 0$,

$$\text{slope of tangent to } f(x) \text{ at } x = \lim_{h \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Finding derivatives this way is tedious but a number of shortcut rules are available. In fact we can use these rules to find the *function* that gives the slope of the tangent to $f(x)$ at *any* point x. This **derivative function** is given the name $\dfrac{dy}{dx}$ or $f'(x)$.

$$\frac{dy}{dx} \quad \text{or} \quad f'(x).$$

$f(x)$	$f'(x)$	
k (a constant)	0	A constant doesn't change!
x^n	nx^{n-1}, for all n	Eg. $f(x) = x^{-3}$ so $f'(x) = -3x^{-4}$
e^x	e^x	
$\ln x$	$\frac{1}{x}$	
$kf(x)$	$kf'(x)$	Eg. $y = 2x^2$ so $\frac{dy}{dx} = 2 \times 2x^1 = 4x$
$f(x) + g(x)$	$f'(x) + g'(x)$	Eg. $f(x) = x^3 + \ln x$ so $f'(x) = 3x^2 + \frac{1}{x}$
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$	**Product Rule**
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$	**Quotient Rule**

Example: Differentiate $f(x) = 2x + 3$.

Solution: $f'(x) = 2 \times 1x^{1-1} + 0 = 2x^0 = 2$. (This should not be a surprise since $f(x)$ is clearly a straight line with slope 2. The solution $f'(x) = 2$ indicates that the tangent has slope 2 for any value of x, as required.)

Example: Differentiate $f(x) = 4x^3 \ln x$.

Solution: This is a product of x^3 and $\ln x$ with a constant multiple of 4. So

$$\begin{aligned} f'(x) &= 4\left(3x^2 \times \ln x + x^3 \times \frac{1}{x}\right) \\ &= 12x^2 \ln x + 4x^2. \end{aligned}$$

Example: Differentiate $y = \dfrac{x+4}{2x+5}$.

Solution: This is a quotient of $x + 4$ and $2x + 5$ so

$$\begin{aligned} \frac{dy}{dx} &= \frac{1 \times (2x+5) - (x+4) \times 2}{(2x+5)^2} \\ &= -\frac{3}{(2x+5)^2} \end{aligned}$$

Exercises

(1) For $f(x) = x^2$ show that $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = 2x$.

(2) Differentiate $y = x^2(2x-1)$ with respect to x by
(a) expanding the RHS first, and
(b) the product rule.

(3) Differentiate the following functions with respect to x:

(a) $y = x^4 + x^{-4} + 4$ (b) $f(x) = \dfrac{1}{x}$

(c) $y = 6x^{-\frac{2}{3}}$ (d) $f(x) = 4x^3 e^x$

(e) $\sqrt{x}(3x-1)$ [recall that $\sqrt{x} = x^{\frac{1}{2}}$]

(f) $y = e^x + (x^4+1)\ln x + 5$ (g) $f(x) = x^5(x^2+6)(x+e^x)$

(h) $y = ax^2 + bx + c$ where a, b and c are constants

(i) $f(x) = a^3 + a^2 b + ab^2 + b^3$ where a and b are constants

(j) $y = \dfrac{1+3x}{2-x}$ (k) $f(x) = \dfrac{x^2-3x+1}{x+2}$

(l) $y = \dfrac{(x+1)e^x}{x}$ (m) $f(x) = \dfrac{\sqrt{x}}{5x+2}$

(n) $y = \dfrac{1}{6x^2+7}$

The Chain Rule

This is the most useful rule of the lot and is based on the following idea:

$$\frac{dy}{dx} = \frac{dy}{\boldsymbol{du}} \times \frac{\boldsymbol{du}}{dx},$$

where u is a function of x that suits you.

Example: $y = e^{3x}$ can't be differentiated by the current rules but it could be done if $u(x) = 3x$ and we apply the chain rule.

$$y = e^u \quad \text{so} \quad \frac{dy}{du} = e^u \quad \text{and} \quad \frac{du}{dx} = 3.$$

Hence

$$\frac{dy}{dx} = e^u \times 3 = 3e^{3x}.$$

Example: Consider $y = (2x+1)^3$. If we expand the brackets we get

$$y = (4x^2+4x+1)(2x+1) = 8x^3 + 12x^2 + 6x + 1$$

and hence

$$\frac{dy}{dx} = 24x^2 + 24x + 6 = 6(4x^2+4x+1) = 6(2x+1)^2.$$

The chain rule is more efficient (especially in cases where the power is higher than 3) if we let $u(x) = 2x + 1$.

$$y = u^3 \quad \text{so} \quad \frac{dy}{du} = 3u^2 \quad \text{and} \quad \frac{du}{dx} = 2.$$

Hence

$$\frac{dy}{dx} = 3u^2 \times 2 = 6(2x+1)^2 \text{ as before.}$$

Exercises

(4) Differentiate $y = (4x-5)^2$ with respect to x by

(a) expanding the RHS first, and
(b) the chain rule.

(5) Repeat question (3)(n) by noting that $\dfrac{1}{6x^2+7} = (6x^2+7)^{-1}$ and using the chain rule.

(6) Differentiate the following functions with respect to x.

(a) $y = e^{4x^2}$ (b) $f(x) = \ln(1-x)$ (c) $y = \sqrt{2x+1}$

(d) $f(x) = \sqrt[3]{x^3 - x^2}$ (e) $y = e^{(x^3+6)^4}$ (f) $y = (2x+1)^{10}$

(g) $f(x) = (x+a)^b$ where a and b are constants (h) $y = \ln(e^x)$

Answers to Exercises

(2) $6x^2 - 2x$

(3) (a) $4x^3 - 4x^{-5}$ (b) $-\dfrac{1}{x^2}$ (c) $-4x^{-\frac{5}{3}}$

(d) $12x^2e^x + 4x^3e^x$ (e) $\dfrac{3x-1}{2\sqrt{x}} + 3\sqrt{x}$ (f) $e^x + 4x^3\ln x + \dfrac{x^4+1}{x}$

(g) $(7x^6 + 30x^4)(x+e^x) + (x^7 + 6x^5)(1+e^x)$ (h) $2ax + b$

(i) 0 (j) $\dfrac{7}{(2-x)^2}$ (k) $\dfrac{x^2+4x-7}{(x+2)^2}$

(l) $\dfrac{e^x(x+x^2-1)}{x^2}$ (m) $\dfrac{2x^{-\frac{1}{2}} - 5x^{\frac{1}{2}}}{2(5x+2)^2}$

(n) $-\dfrac{12x}{(6x^2+7)^2}$

(4) $8(4x-5)$

(5) $-(6x^2+7)^{-2} \times 12x = -\dfrac{12x}{(6x^2+7)^2}$

(6) (a) $8xe^{4x^2}$ (b) $-\dfrac{1}{1-x}$ (c) $\dfrac{1}{\sqrt{2x+1}}$

(d) $\dfrac{3x^2-2x}{3(x^3-x^2)^{\frac{2}{3}}}$ (e) $12x^2(x^3+6)^3e^{(x^3+6)^4}$ (f) $20(2x+1)^9$

(g) $b(x+a)^{b-1}$ (h) 1 (Hardly surprising since $\ln(e^x) = x$.)