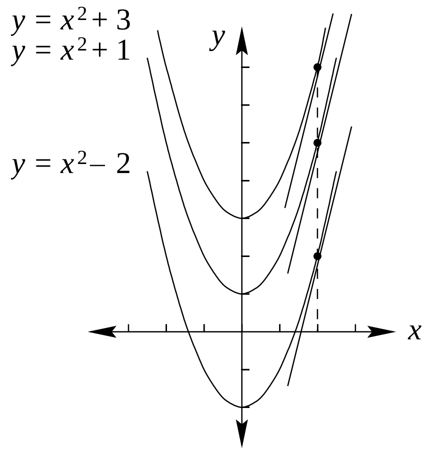
Maths Learning Service: Revision ***Mathematics IA***

Anti-differentiation (Integration)

Anti-differentiation

Anti-differentiation or *integration* is the reverse process to differentiation. For example, if $f'(x) = 2x$, we know that this is the derivative of $f(x) = x^2$. Could there be any other possible answers?

If we shift the parabola $f(x) = x^2$ by sliding it up or down vertically, all the points on the curve will still have the same tangent slopes, i.e. derivatives. For example:



all have the same derivative function, $y' = 2x$, so a general expression for this family of curves would be

$$y = x^2 + c \qquad \text{where } c \text{ is an arbitrary constant (called the integration constant).}$$

Note: Where possible, check your answer by differentiating, remembering that the derivative of a constant, c, is zero.

In mathematical notation, this anti-derivative is written as

$$\int 2x\,dx = x^2 + c$$

The integration symbol "$\int$" is an extended S for "summation". (You will see why in Mathematics IM.)

The "dx" part indicates that the integration is with respect to x. For instance, the integral $\int 2x\,dt$ can not be found, unless x can be rewritten as some function of t.

Examples: (1) If $y' = x$, then $y = \frac{1}{2}x^2 + c$ (Check: $y' = \frac{1}{2} \times 2x + 0 = x$ ✓)

(2) If $y' = x^2$, then $y = \frac{1}{3}x^3 + c$ (Check: $y' = \frac{1}{3} \times 3x^2 + 0 = x^2$ ✓)

(3) $\int x^{-3}dx = -\frac{x^{-2}}{2} + c$ (Check: $\frac{d}{dx}\left(-\frac{x^{-2}}{2} + c\right) = -2 \times -\frac{x^{-3}}{2} + 0 = x^{-3}$ ✓)

(4) $\int x^{\frac{1}{2}}dx = \frac{x^{\frac{3}{2}}}{3/2} + c = \frac{2x^{\frac{3}{2}}}{3} + c$ (Check: $\frac{d}{dx}\left(\frac{2x^{\frac{3}{2}}}{3} + c\right) = \frac{3}{2} \times \frac{2x^{\frac{1}{2}}}{3} + 0 = x^{\frac{1}{2}}$ ✓)

Notice that a pattern emerges which can be summarized in mathematical notation as

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$$

for *any* real number n, except -1.

When $n = -1$ this formula would give $\int x^{-1}dx = \frac{x^0}{0} + c$, which is undefined. However, the integral does exist. Since $\frac{d}{dx}(\ln x) = \frac{1}{x}$ we can say $\int x^{-1}dx = \ln|x| + c$.

As a consequence of other basic rules of differentiation, we also have

$$\begin{aligned} \int kg(x)\, dx &= k\int g(x)\, dx, \text{ where } k \text{ is a constant.} \\ \int (g(x) + h(x))\, dx &= \int g(x)\, dx + \int h(x)\, dx \\ \int e^{ax+b}\, dx &= \frac{1}{a}e^{ax+b} + c, \text{ where } a,\ b \text{ and } c \text{ are constants.} \end{aligned}$$

Examples: (1) $\int \left(x^2 + x\right) dx = \frac{x^3}{3} + \frac{x^2}{2} + c$ (2) $\int e^{3x-2}dx = \frac{1}{3}e^{3x-2} + c$

(3) $\int \left(4x^{\frac{1}{2}} + 3\right) dx = 4 \times \frac{2x^{\frac{3}{2}}}{3} + 3x + c = \frac{8x^{\frac{3}{2}}}{3} + 3x + c$

Exercises

1. Find the following integrals. Check each answer by differentiating.
   (a) $\int x^9 dx$ (b) $\int x^{\frac{1}{4}} dx$ (c) $\int x^{-5} dx$ (d) $\int x^{-\frac{3}{2}} dx$ (e) $\int 1\, dx$
2. Find the following integrals. Check each answer by differentiating.
   (a) $\int \left(2x^{\frac{1}{2}} + \frac{3}{x^2} + 1\right) dx$ (b) $\int \left(1 - 4x + 9x^2\right) dx$ (c) $\int (2x+1)^2\, dx$
   (d) $\int \frac{3}{x^2}\, dx$ (e) $\int e^{7x}\, dx$ (f) $\int e^{-x-1}\, dx$

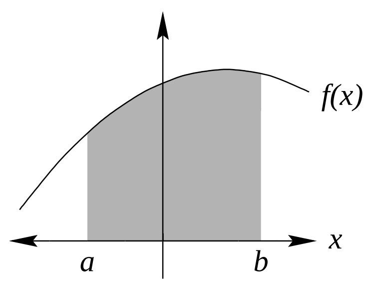
Definite Integration and areas under curves

The *definite integral* $\int_a^b f(x)dx$ is the number $F(b) - F(a)$, where $F(x) = \int f(x)dx$, the antiderivative of $f(x)$.

Example: $\int_0^2 \left(x^2 - x\right) dx = \left[\frac{x^3}{3} - \frac{x^2}{2} + c\right]_0^2 = \left(\frac{8}{3} - \frac{4}{2} + c\right) - \left(\frac{0}{3} - \frac{0}{2} + c\right) = \frac{2}{3}.$

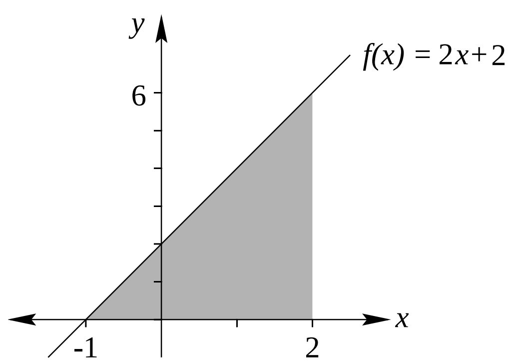
Note: The constant c cancels out in definite integration.

If $f(x) \geq 0$ and continuous in the interval $a \leq x \leq b$ then $\int_a^b f(x)dx$ is the shaded area under the curve between a and b:



Example: $\int_{-1}^2 (2x + 2)dx = \left[x^2 + 2x\right]_{-1}^2 = (4 + 4) - (1 - 2) = 9.$

For this simple curve we can check the area:



The shaded area is $\frac{1}{2} \times 3 \times 6 = 9$ ✓.

The area enclosed between two curves $f(x)$ and $g(x)$, where $f(x) \geq g(x)$ in the interval $a \leq x \leq b$, is given by

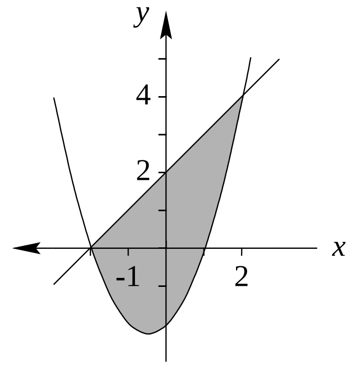
$$\int_a^b f(x)dx - \int_a^b g(x)dx \;=\; \int_a^b (f(x) - g(x))\, dx$$

irrespective of the position of the $x-$*axis.*

Example: Find the area enclosed between $f(x) = x + 2$ and $g(x) = x^2 + x - 2$.

The curves intersect when $x + 2 = x^2 + x - 2$ or $x^2 - 4 = (x - 2)(x + 2) = 0$, ie. when $x = \pm 2$. Since $f(x) \geq g(x)$ in the interval $-2 \leq x \leq 2$ we have

$$\begin{aligned}\int_{-2}^{2}\left(x+2-(x^2+x-2)\right)dx &= \int_{-2}^{2}\left(4-x^2\right)dx\\ &= \left[4x-\frac{x^3}{3}\right]_{-2}^{2}\\ &= \left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)\\ &= 16-\frac{16}{3} = \frac{32}{3}\ \text{units}^2\end{aligned}$$



Exercises

3. Calculate

(a) $\int_{2}^{5} e^x\,dx$ (b) $\int_{-2}^{0} 3x^2\,dx$ (c) $\int_{4}^{9} 2\left(\sqrt{x}-x\right)\,dx$ (d) $\int_{-2}^{-1}\left(2x^3+\frac{1}{x^2}\right)dx$

4. Find the area between the following functions and the $x-$axis for the indicated interval.

(a) $x^{\frac{1}{3}}$, $1\le x\le 8$ (b) $\sqrt{x}-x$, $4\le x\le 9$

5. Find the area between (a) $y=x-2$ and $y=2x-x^2$ (b) $y=x^2$ and $x=y^2$

ANSWERS

1. (a) $\frac{x^{10}}{10}+c$ (b) $\frac{4x^{\frac{5}{4}}}{5}+c$ (c) $-\frac{x^{-4}}{4}+c$ (d) $-2x^{-\frac{1}{2}}+c$ (e) $x+c$

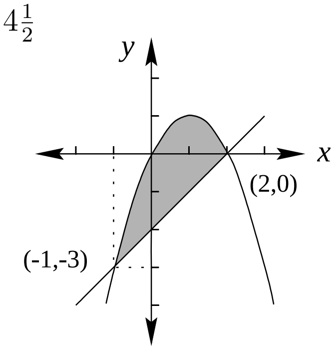
2. (a) $\frac{4}{3}x^{\frac{3}{2}}-\frac{3}{x}+x+c$ (b) $x-2x^2+3x^3+c$ (c) $\frac{4}{3}x^3+2x^2+x+c$

(d) $-\frac{3}{x}+c$ (e) $\frac{1}{7}e^{7x}+c$ (f) $-e^{-x-1}+c$

3. (a) 141.024 (b) 8 (c) $-39\frac{2}{3}$ (d) -7

4. (a) $11\frac{1}{4}$ (b) $19\frac{5}{6}$

5. (a) $4\frac{1}{2}$



(b) $\frac{1}{3}$