MathsTrack

(NOTE Feb 2013: This is the old version of MathsTrack.
New books will be created during 2013 and 2014)

THE UNIVERSITY of ADELAIDE

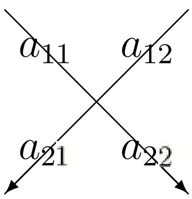
Topic 2

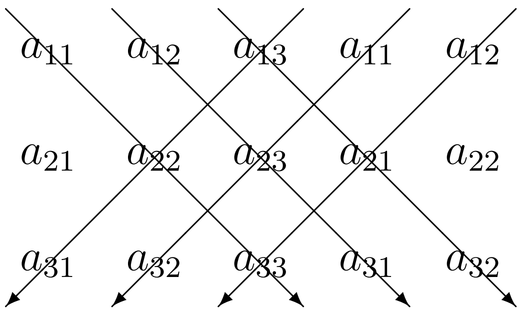
Matrices

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$(A+B)+C = A+(B+C) \qquad h(kA) = (hk)A \qquad (AB)C = A(BC)$$

$$[I|B] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{array}\right] \qquad A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$





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This Topic ...

Matrices[1] were originally introduced as an aid to solving simultaneous linear equations, but now have an important role in many areas of pure and applied mathematics. Today, matrix theory is used in business, economics, statistics, engineering, operations research, biology, chemistry, physics, meterology, etc. Matrices may contain many thousands of numbers, and need to be analysed with computers.

This topic introduces the theory of matrices. For convenience, the examples and exercises in the topic use small matrices, however the ideas are applicable to matrices of any size.

The topic has 2 chapters:

Chapter 1 introduces matrices and their entries. It begins by showing how matrices are related to tables, and gives examples of the different ways matrices and their entries can be described and written down.

Matrix algebra is introduced next. Equality, matrix addition/subtraction, scalar multiplication and matrix multiplication are defined. Examples show how these ideas arise naturally from real life situations, and how matrix algebra grows out of practical applications. The rules for the algebra of matrices are based upon the rules of the algebra of numbers. Some of these rules are introduced, and their connection with the corresponding rules for real numbers is described.

After reading this chapter, you will have a good idea of what matrices are, and will see how your previous knowledge carries across into matrices.

Chapter 2 introduces the inverse of a square matrix. The chapter shows how a system of linear equations can be written as a matrix equation, and how this can be solved using the idea of the inverse of a matrix. Not all matrices have inverses, so the questions of which matrices have inverses and how these inverses are calculated are looked at.

Auhor: Dr Paul Andrew Printed: February 24, 2013

[1] *matrices* is the plural of *matrix*

Contents

Chapter 1

The Algebra of Matrices

1.1 Matrices and their Entries

A matrix is a rectangular array (or pattern) of numbers. Numerical data is frequently organised into tables, and can be represented as a matrix. The theory of matrices often enables this information to be analysed further.

arrays of numbers

Example
The table below shows the number of units of materials and labour needed to manufacture three products in one week.

	Product 1	Product 2	Product 3
Labour	10	12	16
Materials	5	9	7

The matrix representing these data can be written as either

$$\begin{bmatrix} 10 & 12 & 16 \\ 5 & 9 & 7 \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} 10 & 12 & 16 \\ 5 & 9 & 7 \end{pmatrix},$$

with the rectangular array of numbers enclosed inside a pair of large brackets.[1]

When new concepts are introduced in mathematics, we need to define them precisely.

Definition 1.1.1
An $m \times n$ (pronounced 'm by n') **matrix** *is a rectangular array of real numbers having m rows and n columns. The matrix is said to have* **order** $m \times n$. *The numbers in the matrix are called the* **entries** *(sing. entry) of the matrix.* [2]

[1]We will use square brackets in this topic.
[2]Many older texts call these numbers the **entries** of the matrix.

Example

order of a matrix

Four matrices of different orders (or sizes) are given below:

(a) $\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$ is a 2×2 matrix; it has 2 rows and 2 columns.

(b) $\begin{bmatrix} 12 & 56 & 7 \\ 8 & 42 & 5 \end{bmatrix}$ is a 2×3 matrix; having 2 rows and 3 columns.

(c) $\begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 6 & -2 \end{bmatrix}$ has order 3×2.

(d) $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is a general 3×3 matrix. Its entries are not specified and are represented by letters.

In a large general matrix it is impractical to represent each entry by a different letter. Instead, a single uppercase letter (A, B, C, etc) is used to represent the whole matrix, and a lower case letter (a, b, c, etc) with a *double-subscript* is used to represent its entries. For example, a general 2×2 matrix could be represented as

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and a general $m \times n$ matrix could be represented as

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ \vdots & & & & \\ b_{m1} & b_{m2} & b_{m3} & \dots & b_{mn} \end{bmatrix}, \tag{1.1}$$

which is commonly abbreviated to

$$B = [b_{ij}]_{m \times n} \quad \text{or} \quad B = [b_{ij}]. \tag{1.2}$$

In matrix B above, b_{ij} represents the (i, j)-entry located in the i-th row and the j-th column. For example, b_{23} is the (2, 3)-entry in the second row and third column, and b_{m2} is the entry in the m-th row and second column.

Example

matrix entries

(a) If $A = \begin{bmatrix} 1 & 2 \\ 4 & w \end{bmatrix}$, then $a_{11} = 1$, $a_{12} = 2$, $a_{21} = 4$ and $a_{22} = w$.

(b) If $b_{ij} = ij$, then $[b_{ij}]_{2\times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$.

It is common to see matrices whose entries have special patterns. These matrices are often given names related to these patterns.

Example

special matrices

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is a 3×3 matrix and is also called a *square matrix of order 3.*

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a *diagonal matrix of order 3.* Its entries satisfy the condition $a_{ij} = 0$ when $i \neq j$.

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ is an *upper triangular matrix of order 3.* Its entries satisfy the condition $a_{ij} = 0$ when $i > j$.

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ is a *lower triangular matrix of order 3.* Its entries satisfy the condition $a_{ij} = 0$ when $i < j$.

(e) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ is a 1×3 *row matrix* or *row vector.*[3]

(f) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a 3×1 *column matrix* or *column vector.*

Exercise 1.1

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1. The matrix A has order $p \times q$. How many entries are in
   (a) the matrix A?
   (b) the first first row of A?
   (c) the last column of A?

2. Let

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \\ 4 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 6 & x \\ 2 & y & -3 \end{bmatrix}.$$

   Write down each of the following:

   (a) the $(1,2)$-entry of A (b) the $(2,2)$-entry of B (c) a_{32} (d) b_{23}

3. Write down the general 2×3 matrix A with entries a_{ij} in the form (1.1).

4. Write down the 3×1 column vector B with $b_{21} = 2$ and $b_{ij} = 0$ otherwise.

5. Write down the 3×2 matrix C with entries $c_{ij} = i + j$.

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[3]The word *vector* comes from applications of matrices in geometry.

1.2 Equality, Addition and Multiplication by a Scalar

When new objects are introduced in mathematics, we need to specify what makes two objects *equal*. For example, the two rational numbers $\frac{2}{3}$ and $\frac{4}{6}$ are called equal even though they have different representations.

Definition 1.2.1
Two matrices A and B are called **equal** *(written $A = B$) if and only if:* [4]

- *A and B have the same size*
- *corresponding entries are equal.*

If A and B are written in the form $A = [a_{ij}]$ and $B = [b_{ij}]$, introduced in (1.2), then the second condition takes the form:

$$[a_{ij}] = [b_{ij}] \qquad \textit{means} \qquad a_{ij} = b_{ij} \textit{ for all } i \textit{ and } j.$$

matrix equality

Example
If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix},$$

then

(a) A is not equal to B because they have different sizes: A has order 2×2 and B has order 2×3.

(b) B is not equal to C for the same reason.

(c) $A = C$ is true when corresponding entries are equal: $a = 1$, $b = 2$, $c = 3$ and $d = -1$.

a matrix equation

Example
Solve the matrix equation

$$\begin{bmatrix} x-2 & y+3 \\ p+1 & q-1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ -1 & -1 \end{bmatrix}.$$

Answer
As corresponding entries are equal:

(a) $x - 2 = 6 \Rightarrow x = 8$ (b) $y + 3 = 5 \Rightarrow y = 2$

(c) $p + 1 = -1 \Rightarrow p = -2$ (d) $q - 1 = -1 \Rightarrow q = 0$

[4] The phrase *if and only if* is used frequently in mathematics. If P stands for some statement or condition, and Q stands for another, then "P if and only if Q" means: if either of P or Q is true, then so is the other; if either is false, then so is the other. So P and Q are either both true or both false.

The English lawyer-mathematician Arthur Cayley developed rules for adding and multiplying matrices in 1857. These rules correspond to how we organise and manipulate tables of numerical data.

matrix addition

Example
The table below shows the number of washing machines shipped from two factories, F1 and F2, to three warehouses, W1, W2 and W3, during November. This situation is represented by the matrix N.

November	W1	W2	W3
F1	40	50	65
F2	20	35	30

$$\Longrightarrow \quad N = \begin{bmatrix} 40 & 50 & 65 \\ 20 & 35 & 30 \end{bmatrix}$$

Matrix D represents the shipments made during December.

December	W1	W2	W3
F1	50	60	75
F2	30	45	50

$$\Longrightarrow \quad D = \begin{bmatrix} 50 & 60 & 75 \\ 30 & 45 & 50 \end{bmatrix}$$

The total shipment for both months is:

Total	W1	W2	W3
F1	40 + 50	50 + 60	65 + 75
F2	20 + 30	35 + 45	30 + 50

$$\Longrightarrow \quad T = \begin{bmatrix} 90 & 110 & 140 \\ 50 & 80 & 80 \end{bmatrix}$$

This example suggests that it is useful to add matrices by adding corresponding entries:

$$N + D = \begin{bmatrix} 40+50 & 50+60 & 65+75 \\ 20+30 & 35+45 & 30+50 \end{bmatrix} = T.$$

matrix subtraction

Example (continued)
During December the number of washing machine sold is represented by the matrix

$$S = \begin{bmatrix} 46 & 57 & 60 \\ 28 & 42 & 45 \end{bmatrix},$$

so the number of washing machines in the December shipment which were not sold is given by the matrix

$$D - S = \begin{bmatrix} 50 & 60 & 75 \\ 30 & 45 & 50 \end{bmatrix} - \begin{bmatrix} 46 & 57 & 60 \\ 28 & 42 & 45 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 15 \\ 2 & 3 & 5 \end{bmatrix}.$$

This example shows how two matrices may be subtracted.

Definition 1.2.2
If A and B are matrices of the same size, then their **sum** *$A + B$ is the matrix formed by adding corresponding entries. If $A = [a_{ij}]$ and $B = [b_{ij}]$, this means*

$$A + B = [a_{ij} + b_{ij}].$$

adding matrices

Example
If

$$A = \begin{bmatrix} 2 & u \\ 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$$

then

(a) $A + B$ is not defined because A and B have different sizes,

(b) $A + C = \begin{bmatrix} 2+1 & u+2 \\ 1+3 & 4-1 \end{bmatrix} = \begin{bmatrix} 3 & u+2 \\ 4 & 3 \end{bmatrix}$.

a matrix equation

Example
Find a and b, if

$$\begin{bmatrix} a & b \end{bmatrix} + \begin{bmatrix} 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \end{bmatrix},$$

Answer

Add the matrices on the left side to obtain

$$\begin{bmatrix} a+2 & b-1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \end{bmatrix}.$$

As corresponding entries are equal: $a = -1$ and $b = 4$.

When we first learnt to multiply by whole numbers, we learnt that multiplication was *repeated addition*, for example if x is any number then:

$$2x \text{ is } x + x, 3x \text{ is } x + x + x, \text{and so on } \ldots$$

This idea carries over to matrices, for example if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,
then it is natural to write:

$$2A = A + A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix},$$

$$3A = A + 2A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix} = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}, \text{ and so on} \ldots$$

This form of multiplication is called *scalar multiplication*, and it is defined below for all real numbers. The word *scalar* is the the traditional name for a number which multiplies a matrix.

Definition 1.2.3
If A is any matrix and k is any number, then the **scalar multiple** *kA is the matrix obtained from A by multiplying each entry of A by k. If $A = [a_{ij}]$, then*

$$kA = [ka_{ij}] .$$

We can use scalar multiplication to define what the *negative* of a matrix means, and what the *difference* between two matrices is.

Definition 1.2.4
If B is any matrix, then the **negative** *of B is $-B = (-1)B$. If $B = [b_{ij}]$, then*

$$-B = [-b_{ij}]$$

Definition 1.2.5
If A and B are matrices of the same size, then their **difference** *is $A-B = A+(-B)$. If $A = [a_{ij}]$ and $B = [b_{ij}]$, then*

$$A - B = [a_{ij}] + [-b_{ij}] = [a_{ij} - b_{ij}]$$

evaluating expressions

Example
If

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 \\ 3 & -5 \end{bmatrix},$$

find (a) $2A + 3B$ and (b) $2A - 3B$.

Answer

(a) $$2A + 3B = 2\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} + 3\begin{bmatrix} 1 & 4 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 12 \\ 9 & -15 \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ 13 & -9 \end{bmatrix}$$

(b) $$2A - 3B = 2\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 4 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 12 \\ 9 & -15 \end{bmatrix} = \begin{bmatrix} -1 & -12 \\ -5 & 21 \end{bmatrix}$$

a matrix equation

Example
Solve the matrix equation

$$x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} -3 \\ 5 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 6 \end{bmatrix},$$

for the scalars x and y.

Answer
Evaluate the left and right sides separately:[5]

[5] left side = LS, right side = RS

$$\text{LS} = x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2x \\ x \end{bmatrix} + \begin{bmatrix} -3y \\ 5y \end{bmatrix} = \begin{bmatrix} 2x-3y \\ x+5y \end{bmatrix}$$

$$\text{RS} = 2\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 12 \end{bmatrix}$$

As the corresponding entries of

$$\begin{bmatrix} 2x-3y \\ x+5y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 \\ 12 \end{bmatrix}$$

are equal, we need to solve:

$$\begin{aligned} 2x - 3y &= -2 \\ x + 5y &= 12 \end{aligned} \quad ,$$

... with solutions $x = 2$ and $y = 2$.

The number *zero* has a special role in mathematics. It can be a starting value (time = 0), a transitional value (profit = loss = 0), and a reference value (the origin on the real line). Also, 0 is the only number for which

$$x - x = 0 \quad \text{and} \quad x + 0 = 0 + x = x \quad \text{for every number } x.$$

The *zero matrix* has similar properties to the number zero.

Definition 1.2.6
The $m \times n$ matrix with all entries equal to zero is called the $m \times n$ **zero matrix***, and is represented by O (or $O_{m\times n}$ if it is important to emphasise the size). O can also be written in the form $O = [0]$ or $[0]_{m\times n}$, introduced in (1.2).*

zero matrices

Example
Each matrix below is a zero matrix:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

properties of zero matrices

Example
If A is any $m \times n$ matrix, prove that $A - A = O$.

Proof
If $A = [a_{ij}]$, then

$$A - A = [a_{ij}] - [a_{ij}] = [a_{ij} - a_{ij}] = [0]_{m\times n} = O$$

Example
If A is any $m \times n$ matrix, prove that $A + O = A$.

Proof
If $A = [a_{ij}]$, then

$$A + O = [a_{ij}]_{m\times n} + [0]_{m\times n} = [a_{ij} + 0] = [a_{ij}] = A$$

Exercise 1.2

1. If
$$P = \begin{bmatrix} 1 & -5 & 4 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix},$$
find $2P$, $\frac{1}{2}Q$, and $3P - 2Q$.

2. Find a, b, c, and d if
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 3a & 2b \\ c & d \end{bmatrix} = \begin{bmatrix} 8 & -6 \\ -2 & 1 \end{bmatrix}$$

3. Find p and q if
$$3\begin{bmatrix} p \\ q \end{bmatrix} + 2\begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

4. Find u and v if
$$u\begin{bmatrix} 3 \\ 1 \end{bmatrix} + v\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

5. If A is any $m \times n$ matrix, prove that $O + A = A$.

6. If A is any $m \times n$ matrix, prove that $0A = O$.

1.3 Rules of Matrix Algebra (Part I)

Please read Appendix A before starting this section.

Most of the rules that we use for simplifying numerical expressions also apply when simplifying matrix expressions. This means that most of our skills in the algebra of numbers will carry over to the algebra of matrices ... but we need to know which rules we can use and which we can not.

Rules for Matrix Addition

If A, B and C are matrices of the same size, then

(1.3) $(A + B) + C = A + (B + C)$ (*associative rule*)

(1.4) $A + B = B + A$ (*commutative rule*).

These rules are the same as the rules for adding real numbers.

As with real numbers, these rules allow us to:

- write matrix sums without needing to use brackets.
- rearrange the terms in a matrix sum into any order we like.

Rules for Scalar Multiplication

If A, B are matrices of the same size and if h, k are scalars, then

(1.5) $h(A + B) = hA + hB$ (*left distributive rule*).

(1.6) $hA + kA = (h + k)A$ (*right distributive rule*).

(1.7) $h(kA) = (hk)A$ (*associative rule*)

These rules allow us to

- expand brackets
- simplify matrix expressions by combining like terms

in exactly the same way that we do when working with real numbers.

combining like terms

Example

If P and Q are matrices of the same size, simplify

$$P + Q + 3Q + 2P,$$

by combining like terms.

Answer

$$P + Q + 3Q + 2P = 3P + 4Q$$

... can you see where rule (1.6) was used?

expanding brackets

Example
If A and B are matrices of the same size, simplify

$$A + 2B + 3(2A + B)$$

Answer

$$\begin{aligned} A + 2B + 3(2A + B) &= A + 2B + 6A + 3B \\ &= 7A + 5B \end{aligned}$$

... can you see where rules (1.5) and (1.7) were used?

Rules for Matrix Subtraction

If we replace the difference $A - B$ by the sum $A + (-B)$, then all the rules for matrix addition carry over to matrix subtraction. *This is exactly how we work real number expressions that contain differences.*

As with real numbers, these rules allow us to:

- write matrix sums and differences without needing to use brackets.
- rearrange the terms in matrix sums and differences into any order we like.

combining negative like terms

Example
If R and S are matrices of the same size, simplify

$$R - S - 3S - 2R,$$

by combining like terms.
Answer

$$R - S - 3S - 2R = R + (-1)S + (-3)S + (-2)R = -R - 4S$$

... can you see where rule (1.6) was used?

(*There is no need to write down every step in your own answers.*)

The rules for matrix addition and scalar multiplication allow us to solve matrix equations in the same way that we solve real number equations.

matrix equations

Example
Solve the matrix equation below for the unknown matrix X:

$$\begin{bmatrix} 2 & 3 & 9 \\ 3 & 5 & 4 \end{bmatrix} + X = \begin{bmatrix} 4 & 3 & 15 \\ 2 & 3 & 5 \end{bmatrix}$$

Answer
X must be a 2×3 matrix.

$$\begin{aligned} \begin{bmatrix} 2 & 3 & 9 \\ 3 & 5 & 4 \end{bmatrix} + X &= \begin{bmatrix} 4 & 3 & 15 \\ 2 & 3 & 5 \end{bmatrix} \\ \begin{bmatrix} 2 & 3 & 9 \\ 3 & 5 & 4 \end{bmatrix} + X - \begin{bmatrix} 2 & 3 & 9 \\ 3 & 5 & 4 \end{bmatrix} &= \begin{bmatrix} 4 & 3 & 15 \\ 2 & 3 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 9 \\ 3 & 5 & 4 \end{bmatrix} \\ X &= \begin{bmatrix} 2 & 0 & 6 \\ -1 & -2 & 1 \end{bmatrix} \end{aligned}$$

Example
Solve the matrix equation below for the unknown matrix X:

$$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} + 2X = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix}$$

Answer

X must be a 2×2 matrix.

$$\begin{aligned} \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} + 2X &= \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} + 2X - \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} &= \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \\ 2X &= \begin{bmatrix} 2 & 4 \\ 2 & -2 \end{bmatrix} \end{aligned}$$

Multyiply both sides by the scalar $\frac{1}{2}$:

$$\begin{aligned} \frac{1}{2} \times 2X &= \frac{1}{2} \begin{bmatrix} 2 & 4 \\ 2 & -2 \end{bmatrix} \\ X &= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \end{aligned}$$

Note. Here we multiply both sides by $\frac{1}{2}$ rather than divide both sides by 2, because multiplication by scalars has been defined and division of matrices by scalars has not been defined!

(*There is no need to write every step in these answers.*)

Example
If A and B are matrices of the same size, solve the following equation for X:

$$2(X + A) + 3(B - 2X) = 4(A + B)$$

Answer

$$\begin{aligned} 2(X + A) + 3(B - 2X) &= 4(A + B) \\ 2X + 2A + 3B - 6X &= 4A + 4B \\ -4X &= 2A + B \\ X &= -\frac{1}{4}(2A + B) \end{aligned}$$

Note. The answer can't be written as $-\frac{2A+B}{4}$ because division of matrices by scalars is not defined.

We can prove that rules of matrix algebra are true by using the rules for real numbers.

two proofs

Example
Prove that the associative rule for matrix addition is true.

Answer
Let A, B and C be an $m \times n$ matrices, then

$$(A+B)+C = ([a_{ij}]+[b_{ij}])+[c_{ij}] = [(a_{ij}+b_{ij})+c_{ij}]$$

and

$$A+(B+C) = [a_{ij}]+([b_{ij}]+[c_{ij}]) = [a_{ij}+(b_{ij}+c_{ij})].$$

By the associative rule for addition of real numbers:

$$(a_{ij}+b_{ij})+c_{ij} = (a_{ij}+b_{ij})+c_{ij} \qquad \text{for each subscript } i \text{ and } j.$$

This shows that

$$(A+B)+C = A+(B+C).$$

Example
Prove that the associative rule for scalar multiplication is true.

Answer
Let h and k be scalars, and let A be an $m \times n$ matrix, then

$$h(kA) = h[ka_{ij}] = [h(ka_{ij})]$$

and

$$(hk)A = hk[a_{ij}] = [(hk)a_{ij}].$$

By the associative rule for multiplication of real numbers:

$$h(ka_{ij}) = (hk)a_{ij} \qquad \text{for each subscript } i \text{ and } j.$$

This shows that

$$h(kA) = (hk)A.$$

Exercise 1.3

1. If L and M are matrices of the same size, simplify.
   (a) L + M + 3L - 2M + L
   (b) 2(3L + M) - 2(M - 2L)
2. Solve the matrix equation

$$2(\begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} + 2X) - (X - 2\begin{bmatrix} 2 & 8 \\ 2 & -3 \end{bmatrix}) = O$$

3. If R and S are matrices of the same size, solve the following equation for T:

$$2(R + S + T) - (2R - 3S + T) + 3(S - T) = O$$

4. Prove that the commutative rule for matrix addition is true.

1.4 Matrix Multiplication

Matrix multiplication is more complicated than matrix addition and scalar multiplication, but it is very useful.

Matrix multiplication is performed by multiplying the rows of the first matrix by the columns of the second matrix: the product of

$$\text{row matrix} \begin{bmatrix} a & b & c \end{bmatrix} \quad \text{and} \quad \text{column matrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

is the 1×1 matrix $[ad + be + cf]$.

row × column

Example

$$\begin{bmatrix} 2 & 5 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = [2 \times 1 + 5 \times 4 + 7 \times 2] = [36]$$

When the first matrix has more than one row, each row is multiplied by the column matrix and the result is recorded in a separate row.

2 rows × column

Example

$$\begin{bmatrix} 2 & 5 & 7 \\ -1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \times 1 + 5 \times 4 + 7 \times 2 \\ (-1) \times 1 + 3 \times 4 + 2 \times 2 \end{bmatrix} = \begin{bmatrix} 36 \\ 15 \end{bmatrix}$$

Tickets × Price

Example

A community theatre held evening performances of a play on Fridays and Saturdays, and tickets were \$30 normal price, \$15 concession and \$10 children. The table below shows the number of tickets sold in the first week.

Tickets	Normal	Concession	Children
Friday	250	100	50
Saturday	200	150	40

The ticket matrix is

$$T = \begin{bmatrix} 250 & 100 & 50 \\ 200 & 150 & 40 \end{bmatrix}$$

and the price matrix is

$$P = \begin{bmatrix} 30 \\ 15 \\ 10 \end{bmatrix}.$$

• The revenue (in \$) for Friday was $250 \times 30 + 100 \times 15 + 50 \times 10 = 9500$. This can also be written as

$$FP = \begin{bmatrix} 250 & 100 & 50 \end{bmatrix} \begin{bmatrix} 30 \\ 15 \\ 10 \end{bmatrix} = [9500],$$

where $F = \begin{bmatrix} 250 & 100 & 50 \end{bmatrix}$ is the Friday ticket matrix and P is the price matrix.

• The revenue (in \$) for Saturday was $200 \times 30 + 150 \times 15 + 40 \times 10 = 8650$. This can also be written as

$$SP = \begin{bmatrix} 200 & 150 & 40 \end{bmatrix} \begin{bmatrix} 30 \\ 15 \\ 10 \end{bmatrix} = [8650],$$

where $S = \begin{bmatrix} 200 & 150 & 40 \end{bmatrix}$ is the Saturday ticket matrix and P is the price matrix.

We can combine these calculations into a single matrix calculation to obtain the Revenue matrix R:

$$R = TP = \begin{bmatrix} 250 & 100 & 50 \\ 200 & 150 & 40 \end{bmatrix} \begin{bmatrix} 30 \\ 15 \\ 10 \end{bmatrix} = \begin{bmatrix} 9500 \\ 8650 \end{bmatrix}.$$

Although this matrix calculation does not give us any new information, it allows us to organise our work in a neat compact intuitive way using the formula Revenue = Tickets × Price. *This is of practical importance when handling large sets of numbers in computers.*

The price matrix P was chosen to be a column vector. This choice conforms with the definition of matrix multiplication (see below). In practice you will need to decide how information should be represented in matrices in order that matrices can be multiplied.

The general rule for multiplying matrices is given by:

Definition 1.4.1
If A is an $m \times n$ matrix and B is an $n \times k$ matrix, then the **product** *AB of A and B is the $m \times k$ matrix whose (i,j)-entry is calculated as follows:*

multiply each entry of *row i* in A by the corresponding entry of *column j* in B, and add the results.

In other words,

$$AB = \begin{bmatrix} \textit{row 1 of } A \times \textit{col 1 of } B & \textit{row 1 of } A \times \textit{col 2 of } B & \textit{row 1 of } A \times \textit{col 3 of } B & \dots \\ \textit{row 2 of } A \times \textit{col 1 of } B & \textit{row 2 of } A \times \textit{col 2 of } B & \textit{row 2 of } A \times \textit{col 3 of } B & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

If A and B are written in the form $A = [a_{ij}]$ and $B = [b_{ij}]$, then this means $AB = C = [c_{ij}]$ where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} \text{ for all } i \text{ and } j.$$

Example

multiplying matrices

If $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then

$$(1)\ AB = \begin{bmatrix} \text{row 1 of } A \times \text{col 1 of } B & \text{row 1 of } A \times \text{col 2 of } B \\ \text{row 2 of } A \times \text{col 1 of } B & \text{row 2 of } A \times \text{col 2 of } B \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 4 + 1 \times 1 + 3 \times 0 & 2 \times 5 + 1 \times 2 + 3 \times 3 \\ (-1) \times 4 + 4 \times 1 + 5 \times 0 & (-1) \times 5 + 4 \times 2 + 5 \times 3 \end{bmatrix} = \begin{bmatrix} 9 & 21 \\ 0 & 18 \end{bmatrix}$$

$$(2)\ BC = \begin{bmatrix} \text{row 1 of } B \times \text{col 1 of } C & \text{row 1 of } B \times \text{col 2 of } C \\ \text{row 2 of } B \times \text{col 1 of } C & \text{row 2 of } B \times \text{col 2 of } C \\ \text{row 3 of } B \times \text{col 1 of } C & \text{row 3 of } B \times \text{col 2 of } C \end{bmatrix}$$

$$= \begin{bmatrix} 4 \times 1 + 5 \times 3 & 4 \times 2 + 5 \times 4 \\ 1 \times 1 + 2 \times 3 & 1 \times 2 + 2 \times 4 \\ 0 \times 1 + 3 \times 3 & 0 \times 2 + 3 \times 4 \end{bmatrix} = \begin{bmatrix} 19 & 28 \\ 7 & 10 \\ 9 & 12 \end{bmatrix}$$

(3) AC is not possible. Matrix multiplication is not defined in this case as the number of entries in a row of A is not equal to the number of entries in a column of B.

Example (3) above highlights an important point. In order to multiply two matrices, *the number of columns in the first matrix must equal the number of rows in the second matrix.* If A is an $m \times n$ matrix and B is an $n \times k$ matrix, then the product AB is an $m \times k$ matrix, that is

$$A_{m\times n}B_{n\times k} = (AB)_{m\times k}.$$

Exercise 1.4

1. If $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, calculate:

   (a) OA (b) AO (c) IA (d) AI
   (e) AB (f) BA (g) $A(BC)$ (h) $(AB)C$
   (i) $A(B + C)$ (j) $AB + BC$

2. What is the (2, 2)-entry in the product:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

3. If possible, calculate.

(a) $\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \end{bmatrix}$

1.5 Rules of Matrix Algebra (Part II)

Please read Appendix A before starting this section.

Most - but not all - of the rules used when multiplying numbers carry over to matrix multiplication.

The Associative Rule for Matrix Multiplication

If A, B and C are matrices which can be multiplied, then

$$(AB)C = A(BC)$$

As with real numbers, this rule allows us to *write matrix products without needing to use brackets.*

The Commutative Rule for Matrix Multiplication

If A and B can be multiplied, then

$$AB \neq BA \ (\textit{except in special cases})$$

counter-examples

Example

If $A = \begin{bmatrix} 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then

(1) $AB = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = [17]$

$\ldots$ and $BA = \begin{bmatrix} 5 \\ 7 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 14 & 7 \end{bmatrix}$.

(2) BC is not defined: B has order 2×1 and C has order 2×2.

$\ldots$ however $CB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 19 \\ 43 \end{bmatrix}$.

(3) $AC = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 8 \end{bmatrix}$

$\ldots$ but CA is not defined: C has order 2×2 and A has order 1×2.

The Distributive Rules for Matrix Multiplication *over* Addition

If A, B and C are matrices, then

(1.8) $\quad A(B + C) = AB + AC \quad$ (left distribution over addition)

(1.9) $\quad (B + C)A = BA + CA \quad$ (right distribution over addition), *(provided these sums and products make sense).*

The distribution rules are also true when there are more than two terms inside the brackets.

These rules allow us to

- expand brackets, and
- combine like terms together.

simplifying expressions

Example
To simplify an expression like

$$2A(A+B) - 3AB + 4B(A-B)$$

we first expand brackets, then combine like terms

$$\begin{aligned} 2A(A+B) - 3AB + 4B(A-B) &= 2AA + 2AB - 3AB + 4BA - 4BB \\ &= 2AA - AB + 4BA - 4BB \end{aligned}$$

Notice that $2AB$ and $(-3)AB$ are like terms which can be combined, but that $2AB$ and $4BA$ are not like terms. This is because the commutative rule is not generally true.

Rules for products of Scalar Multiples of Matrices

If the product A and B is defined, and if h and k are scalars, then

$$(hA)(kB) = hkAB. \tag{1.10}$$

This rule helps us to simplify expressions by combining like terms, just as we do when we work with real numbers - always remembering that the 'commutative rule' is not true for matrix multiplication.

like terms

Example
To simplify an expression like

$$2A(2A+B) - 3AB + 4B(2A-B)$$

we expand brackets, then combine like terms

$$\begin{aligned} 2A(2A+B) - 3AB + 4B(2A-B) &= 4AA + 2AB - 3AB + 8BA - 4BB \\ &= 4AA - AB + 8BA - 4BB \end{aligned}$$

... can you see where rule (1.10) was used?

Exercise 1.5

Expand, then simplify the following expressions

(a) $(A+B)(A-B)$

(b) $(A+B)(A+B) - (A-B)(A-B)$

1.6 The Identity Matrices

The *identity matrices* are similar to the number 1 in ordinary arithmetic.

Definition 1.6.1
The **identity matrix** *of order n is the square matrix of order n with all diagonal entries equal to 1 and all other entries equal to 0, and is represented by I (or I_n if it is important to emphasise the size).*

I can also be written in the form $I = [\delta_{ij}]$ or $[\delta_{ij}]_n$, introduced in (1.2), where $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$.[6]

the identity matrices

Example
Each matrix below is an identity matrix:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{etc.}$$

The identity matrices are similar to the number 1: if A is a $m \times n$ matrix, then

$$I_m A = A \text{ and } AI_n = A.$$

multiplying by an identity

Example
(1) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then:

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$$

and

$$AI = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A.$$

... *check these calculations.*

(2) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, then:

$$I_2 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = A$$

and

$$AI_3 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = A.$$

... *check these calculations as well.*

---

[6] δ is the Greek letter *delta*

1.7 Powers of Square Matrices

Matrix multiplication does not satisfy the commutative rule, so the order of multiplication of two matrices is important. However, there is one special case when the order is not important. This is when powers of the same square matrix are multiplied together.

Definition 1.7.1
If A is a square matrix of order n, and k is a positive integer, then

$$A^k = \underbrace{AA \ \dots \ A}_{k \text{ factors}}.$$

You can see that if r and s are positive integers, then

$$A^r A^s = A^{r+s} = A^s A^r,$$

... so A^r and A^s commute.

matrix powers

Example
If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then

$$A^2 = AA = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}, \qquad A^3 = AA^2 = \begin{bmatrix} 37 & 54 \\ 81 & 118 \end{bmatrix}$$

$$A^4 = AA^3 = \begin{bmatrix} 199 & 290 \\ 435 & 634 \end{bmatrix}, \quad \text{etc} \dots$$

Matrix powers are laborious to calculate by hand and are often calculated with special software packages such as MATLAB. Powers of matrices that satisfy polynomial equations can be found more easily.

polynomial equation

Example
Show that the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

satisfies the polynomial equation $A^2 - 5A - 2I = O$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Answer

$$A^2 - 5A - 2I = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

matrix powers

Example

If the square matrix A satisfies the polynomial equation $A^2 - 5A - 2I = 0$, express A^3 and A^4 in the form $rA + sI$, where r and s are scalars.

Answer

As $A^2 - 5A - 2I = 0$, we can write $A^2 = 5A + 2I$, so

$$A^3 = AA^2 = A(5A + 2I) = 5A^2 + 2A = 5(5A + 2I) + 2A = 27A + 10I,$$

and

$$A^4 = AA^3 = A(27A + 10I) = 27A^2 + 10A = 27(5A + 2I) + 10A = 145A + 54I.$$

Exercise 1.7

1. Show that $A^2 = O$ when

$$A = \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}.$$

2. Show that $B^2 + 3B - 4I = O$ when

$$B = \begin{bmatrix} -1 & 0 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

3. If the square matrix B satisfies the equation $B^2 - 3B - 4I = O$, express B^2, B^3 and B^4 in the form $pB + qI$, where p and q are scalars.

4. Show that $A^3 = A$ when

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}.$$

Explain why you can now write down the matrix A^{27} and A^{31}.

1.8 The Transpose of a Matrix

So far we have looked at matrix addition, scalar multiplication and matrix multiplication. These were based on the operations of addition and multiplication on real numbers. The next operation of finding the *transpose* of a matrix has no counterpart in real numbers.

Definition 1.8.1
If A is an $n \times m$ matrix, then the **transpose** *of A is the $m \times n$ matrix formed by interchanging the rows and columns of A, so that the first row of A is the first column of A^t, the second row of A is the second column of A^t, and so on.*

If $A = [a_{ij}]$, this means $A^t = [a_{ji}]$.

taking transposes

Example

(a) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then $A^t = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$.

(b) If $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, then $B^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^t = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

(d) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^t = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$.

(e) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^t = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Note. Column vectors are frequently represented as transposed row vectors in printed materials (see (e) above) in order to save space and because it's easier to type.

Properties of Matrix transposes

- *If A is a $m \times n$ matrix, then A^t is an $n \times m$ matrix*
- *If A is any matrix, then $(A^t)^t = A$*
- *If A and B are matrices of the same size, then $(A + B)^t = A^t + B^t$*
- *If A is an $m \times n$ matrix and B is an $n \times k$ matrix,then $(AB)^t = B^t A^t$*

Exercise 1.8

1. Evaluate (a) $\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 3 \end{bmatrix}^t$ (b) $\begin{bmatrix} 1 & 2 \end{bmatrix}^t \begin{bmatrix} -1 & 3 \end{bmatrix}$
2. Show that $(AB)^t = B^t A^t$ when

$$A = \begin{bmatrix} 1 & -2 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}.$$

Chapter 2

Inverses of Square Matrices

2.1 Systems of Linear Equations

The graph of the equation $ax + by = c$ is a straight line, so it is natural to call this equation a *linear equation* in x and y. The word *linear* is also used when there are more than two variables.

Many practical problems can be reduced to solving a *system of linear equations.*

Definition 2.1.1
An equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

is called a **linear equation** *in the n variables x_1, x_2, ..., x_n. The numbers a_1, a_2, ..., a_n are called the* **coefficients** *of x_1, x_2, ..., x_n respectively, and b is called the* **constant term** *of the equation.*

A collection of linear equations in the variables x_1, x_2, ..., x_n is called a **system of linear equations** *in these variables.*

a system of linear equations

Example
The three equations in x_1, x_2 and x_3

$$\begin{array}{rcr} 3x_1 + 4x_2 + x_3 & = & 1 \\ 2x_1 + 3x_2 \phantom{{}+x_3} & = & 4 \\ 4x_1 + 3x_2 - x_3 & = & -2 \end{array}$$

is a system of linear equations because each equation is a linear equation.

Note. Although the second equation has two variables, we can think of it as a linear equation in x_1, x_2 and x_3 with the coefficient of x_3 equal to 0. In the third equation, the coefficient of x_3 is -1.

system of non-linear equations

Example
The pair of equations

$$\begin{aligned} 4x_1^2 + 2x_2 &= 4 \\ 2\sqrt{x_1} - 3x_2 &= 2 \end{aligned}$$

is also called a system of equations, but in this case the equations are non-linear: each variable in a linear equation must occur to the first power only.

A system of linear equations can be expressed as a single matrix equation $AX = B$, which may be solved using matrix ideas.

a matrix equation

Example
The system of equations

$$\begin{aligned} 3x_1 + 4x_2 + x_3 &= 1 \\ 2x_1 + 3x_2 &= 4 \\ 4x_1 + 3x_2 - x_3 &= -2 \end{aligned}$$

can be written as the matrix equation

$$\begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 0 \\ 4 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix},$$

that is, in the form $AX = B$, where

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 3 & 0 \\ 4 & 3 & -1 \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix},$$

... which can be solved for the unknown matrix X.

A general system of m equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad \quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

can always be represented as a matrix equation of the form

$$AX = B$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Here the matrix A is called the *coefficient matrix* of the system of equations.

Exercise 2.1

Write the following systems of equations in matrix form.

(a)
$$\begin{aligned} 3x_1 - x_2 &= 1 \\ 2x_1 + 3x_2 &= 4 \end{aligned}$$

(b)
$$\begin{aligned} 3x_1 + 4x_2 - 5x_3 &= 1 \\ 2x_1 - x_2 + 3x_3 &= 0 \end{aligned}$$

(c)
$$\begin{aligned} 4x_2 + x_3 &= 1 \\ 2x_1 - x_3 &= -1 \\ 4x_1 + 3x_2 &= 2 \end{aligned}$$

2.2 The Inverse of a Square Matrix

The matrix equation $AX = B$ is very similiar to the linear equation $ax = b$, where a and b are numbers. This linear equation is solved by multiplying both sides by $1/a = a^{-1}$, the reciprocal or inverse of a:

$$\begin{aligned} ax &= b \\ a^{-1}ax &= a^{-1}b \\ 1x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

The same technique can be used for solving matrix equations of the form $AX = B$ ... once the *inverse* of a matrix is defined

If a is any number, then its inverse is that special number p for which $pa = ap = 1$. For example, the inverse of 2 is 0.5 as $0.5 \times 2 = 2 \times 0.5 = 1$. We traditionally represent the inverse of a by the special symbol a^{-1}, for example $2^{-1} = 0.5$. These same ideas can carried over to matrices.

In matrix theory, *identity matrices* are used to define inverses of square matrices.

Definition 2.2.1
Let A and P be $n \times n$ matrices. If

$$PA = AP = I, \tag{2.1}$$

where I is the $n \times n$ identity matrix, then P is called the **inverse** *of A and is represented by the special symbol A^{-1}. So*

$$A^{-1}A = AA^{-1} = I. \tag{2.2}$$

If A has an inverse, then A is said to be **invertible**.

You can see from this definition that if P is the inverse of A, *then A is the inverse of P*. We can write this in symbols as $P^{-1} = A$ or as

$$(A^{-1})^{-1} = A. \tag{2.3}$$

an invertible matrix

Example
Show that $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

Answer
If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $P = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$, then

$$PA = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

and

$$AP = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

so P is the inverse of A ...and A is the inverse of P.

If A has an inverse, then the matrix equation $AX = B$ can be solved as follows:

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ IX &= A^{-1}B \qquad \ldots \text{ as } A^{-1}A = I \text{ in (2.2)} \\ X &= A^{-1}B \end{aligned}$$

solving a matrix equation

Example
Write the pair of simultaneous equations

$$\begin{aligned} 2x + 5y &= 7 \\ x + 3y &= 4 \end{aligned}$$

as a matrix equation, then solve this matrix equation.
Answer
The matrix equation is

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}.$$

We saw in the previous example that

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix},$$

so

$$\begin{aligned} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \end{bmatrix} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

The matrix equation $AX = B$ can always be solved if A has an inverse ...but not all square matrices are invertible (ie. have inverses).

a non-invertible matrix

Example
The pair of simultaneous equations

$$\begin{aligned} 2x + 5y &= 7 \\ 2x + 5y &= 6 \end{aligned}$$

does not have a solution, so the matrix equation

$$\begin{bmatrix} 2 & 5 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

does not have a solution. This means that the coefficient matrix

$$\begin{bmatrix} 2 & 5 \\ 2 & 5 \end{bmatrix}$$

does not have an inverse, and so is not invertible.

If a and b are numbers, then the reciprocal or inverse of the product ab is

$$(ab)^{-1} = a^{-1}b^{-1}$$

...however this is not true for matrices.

Example

inverse of matrix product

If A and B are $n \times n$ matrices, show that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Answer

We are asked to show that the inverse of AB is $B^{-1}A^{-1}$. To do this we need to show that condition (2.2) in Definition 2.2.1 is satisfied.

It can be confusing when a problem uses the same letters that are in a definition. The way to overcome this is to express the definition in words. In this case, condition (2.2) says:
(1) when a matrix is multiplied on the the left by its inverse, the result is the identity matrix,
(2) when a matrix is multiplied on the right by its inverse, result is the identity matrix again.

... we need to check that (1) and (2) are true for AB and $B^{-1}A^{-1}$.

(1) Multiplying AB on the left by $B^{-1}A^{-1}$:

$$\begin{aligned} (B^{-1}A^{-1})(AB) &= B^{-1}A^{-1}AB && \text{... as there's no need to use brackets} \\ &= B^{-1}IB && \text{... as } A^{-1}A = I \\ &= BB^{-1} && \text{... as } IB = B \\ &= I && \text{... as } BB^{-1} = I \end{aligned}$$

(2) Multiplying AB on the right by $B^{-1}A^{-1}$:

$$\begin{aligned} (AB)(B^{-1}A^{-1}) &= ABB^{-1}A^{-1} && \text{... as there's no need to use brackets} \\ &= AIA^{-1} && \text{... as } BB^{-1} = I \\ &= AA^{-1} && \text{... as } AI = A \\ &= I && \text{... as } AA^{-1} = I \end{aligned}$$

This shows that $(AB)^{-1} = B^{-1}A^{-1}$.

Note. $(AB)^{-1} = B^{-1}A^{-1} \neq A^{-1}B^{-1}$ *as the commutative law is not true for matrix multiplication except in special cases.*

Exercise 2.2

1. Show that $V = \frac{1}{3}\begin{bmatrix} 12 & -13 & -7 \\ -3 & 5 & 2 \\ -3 & 2 & 2 \end{bmatrix}$ is the inverse of $U = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & -1 \\ 3 & 5 & 7 \end{bmatrix}$.

2. Show that $\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$

3. Use your answer to question 2 to solve the following:

(a) The simultaneous equations:

$$\begin{aligned} 3x + 2y &= 4 \\ 4x + 3y &= 7 \end{aligned}$$

(b) The matrix equation $AX = B$, when

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 4 & -2 \end{bmatrix}.$$

(c) The matrix equation $PX = Q$, when

$$P = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & -1 & 2 \\ -2 & 1 & -1 \end{bmatrix}.$$

4. A is a square matrix that satisfies $A^2 - 3A + I = O$, where I is the identity matrix. Show that A has an inverse.

5. If A, B and C are invertible square matrices of order n, show that ABC is invertible and the inverse is $C^{-1}B^{-1}A^{-1}$.

6. If A is an invertible $n \times n$ matrix, then the negative integer powers of A can be defined as

$$A^{-k} = \underbrace{A^{-1}A^{-1} \ \ldots \ A^{-1}}_{k \text{ factors}},$$

where k is a positive integer. Show that $(A^2)^{-1} = A^{-2}$.

Note. If A^0 is defined to be the identity matrix I_n, the power rules $A^rA^s = A^{r+s}$ and $(A^r)^s = A^{rs}$ are true for all integers r and s.

2.3 Inverses of 2×2 Matrices

The matrix equation $AX = B$ can be solved when A is invertible. It is important to decide which square matrices have inverses and how to calculate these inverses.

Theorem
If A is a 2×2 matrix, then A is invertible if and only if [1]

$$a_{11}a_{22} - a_{12}a_{21} \neq 0, \tag{2.4}$$

and, when this condition is true, the inverse is

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}. \tag{2.5}$$

invertible matrix

Example
The matrix

$$\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$$

has an inverse because $3 \times 4 - 5 \times 2 = 2 \neq 0$. The inverse is

$$\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}$$

invertibility

Example
For what values of k is the matrix

$$T = \begin{bmatrix} 3 & k \\ 2 & 4 \end{bmatrix}$$

invertible?
Answer
T is invertible if and only if $3 \times 4 - k \times 2 \neq 0$, that is if and only if $k \neq 6$.

The number $a_{11}a_{22} - a_{12}a_{21}$ is very important, as it tells us when A is invertible.

Definition 2.3.1
If A is a 2×2 matrix, then the number $a_{11}a_{22} - a_{12}a_{21}$ is called the **determinant** *of A. It is represented by the special symbols* $\det A$ *and* $|A|$.[2]

Properties of Determinants

- *If A is a 2×2 matrix, then A is invertible if and only if* $\det A \neq 0$
- *If A and B are 2×2 matrices, then* $\det(AB) = \det A \det B$

[1] See page 5, footnote 1
[2] We will use the symbol $\det A$ in this topic.

Example

matrix product

Let $A = \det \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$ and $B = \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $\det A = 2$ and $\det B = -2$.

As $AB = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 18 & 26 \\ 14 & 20 \end{bmatrix}$, we can see that $\det(AB) = -4$.

This is an example of $\det(AB) = \det A \det B$.

Example

matrix powers

Let A be a 2×2 matrix. Show that $\det(A^2) = (\det A)^2$.

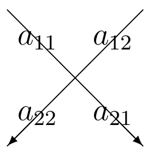
Answer

As $A^2 = AA$, $\det(A^2) = \det AA = \det A \det A = (\det A)^2$.

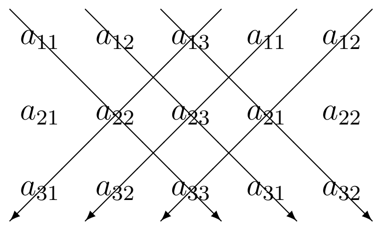
Determinants can also be defined for square matrices of order n, and these have the same properties as the determinants of square matrices of order 2. The formula for a determinant of order 3:

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

You can avoid memorising this formula by using the pictures below:



(a)



(b)

The formula for a 2×2 determinant is obtained from diagram (a) by multiplying the terms on each arrow together, then subtracting those on the arrows going from left to right from those on the arrow going from right to left. The formula for a 3×3 determinant is obtained from diagram (b) similarly.

The formula for a general $n \times n$ determinant can be obatined by writing the matrix down twice, with the copy along side the original as in (b), drawing n arrows going from left to right and n arrows going from right to left as in (b), then continuing as in the 3×3 case.

There is a general formula for the inverse of a square matrix of order n, but the calculation takes a lot of time and a more efficient method is shown in the next section.

Exercise 2.3

1. What is the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

2. Let A be a matrix of order 2 with inverse A^{-1}. Use the properties of determinants to show that $\det(A^{-1}) = 1/\det A$

3. Let A be a matrix of order 2 for which $A^2 = 0$. Use the properties of determinants to show that A does not have an inverse.

4. Evaluate $\det A$, where

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

Is A invertible?

2.4 Calculating the Inverse of an $n \times n$ Matrix

The most efficient way of calculating the inverse of a square matrix is to use *elemenatry row operations.*

This is best shown by example on a 3×3 matrix: To find the inverse of

$$A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix},$$

construct the *augmented-matrix* $[A|I]$ from A and the identity matrix I of order 3:

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right],$$

then use elementary row operations to convert this into the form $[I|B]$, having the identity matrix of order 3 in the first three columns:

$$[I|B] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{array}\right].$$

If you can do this, then the matrix B is the inverse of A. If you can not, then A does not have an inverse.

There are three elementary row operations which can used for this procedure.

Elementary Row Operations
(1) Interchange two rows.
(2) Multiply or divide one row by a non-zero number.
(3) Add a multiple of one row to another row.

To keep track of the our operations, we use the notation:

(1) $R_i \rightleftarrows R_j$ to mean *interchange row i and row j.*
(2) $R_i \to cR_i$ to mean *replace row i by row i multiplied by c.*
(3) $R_j \to R_j + cR_i$ to mean *to add row i multiplied by c to row j.*

calculating the inverse matrix

Example
To change the augmented matrix

$$\left[\begin{array}{ccc|ccc} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

into the form $[I|B]$

- interchange row 1 and row 2, so that the (1, 1)-entry becomes 1.

$$\left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 0 & 1 & 0 \\ 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \quad R_1 \rightleftarrows R_2$$

• subtract 2×row 1 from row 2, and also row 1 from row 3, so that the first entries in rows 2 and 3 become zeros.

$$\left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

• ...continuing we have

$$\left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & -1/2 \end{array}\right] \quad \begin{array}{l} R_2 \to -R_2 \\ R_3 \to R_3/(-2) \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -3/2 \\ 0 & 0 & 1 & 1/2 & -1/2 & -1/2 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 + R_3 \\ R_2 \to R_2 + 3R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/2 & -3/2 & 11/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -3/2 \\ 0 & 0 & 1 & 1/2 & -1/2 & -1/2 \end{array}\right] \quad R_1 \to R_1 - 4R_2$$

So the inverse of A is

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -3 & 11 \\ 1 & 1 & -3 \\ 1 & -1 & -1 \end{bmatrix}$$

Some square matrices are not invertible.

non-invertible matrix

Example
If we try to find the inverse of

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

using the same method, we will end up with a matrix like:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -3/5 & -1/5 & 0 \\ 0 & 0 & 0 & 3/5 & 1/5 & 1 \end{array}\right].$$

The row of zeros in the first part of this matrix shows that it is impossible to obtain a matrix in the form $[I|B]$ where I is the identity matrix of order 3. This implies that A does not have an inverse.

Exercise 2.4

If possible, find the inverse of the following matrices using elementary row transformations.

(a) $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

Appendix A

The Algebra of Numbers

Many of the rules that we use to evaluate or simplify real number expressions can also be used with matrices. These rules are explained below.

Associative Rules for Addition and Multiplication

The associative rules show that if three numbers are added or multiplied, then it doesn't matter how you do this, the answer is always the same.

If a, b and c are real numbers, then

(A.1) $$(a+b)+c = a+(b+c) \qquad (\textit{addition})$$

(A.2) $$(ab)c = a(bc) \qquad (\textit{multiplication})$$

combining three numbers

Example

When three numbers are added or multiplied, there is more than one way of doing this:

$$\text{adding 2, 3 and 4} \Rightarrow \begin{cases} (2+3)+4 = 5+4 = 9 \\ 2+(3+4) = 2+7 = 9 \end{cases}$$

$$\text{multiplying 2, 3 and 4} \Rightarrow \begin{cases} (2\times 3)\times 4 = 6\times 4 = 24 \\ 2\times(3\times 4) = 2\times 12 = 24 \end{cases}$$

The answer is the same in each case, and this is what the associative rules for addition and multiplication describe.

As there is no difference in the answers, there is no need to use brackets when adding or multiplying 3 numbers together ... we can just write $2+3+4$ and $2\times 3\times 4$.

The associative rules show that there is no need to use brackets in sums or products of three numbers. This is also true for sums and products with any number of terms.

We use the associative rules every time we write sums and products without using brackets.

Example

writing without brackets

We don't usually use brackets when writing:

- sums like $2x + 3y + 4x + 5z + 7y$
- products like $x^2y^3x^4z^5y^7$

Commutative Rules for Addition and Multiplication

The commutative rules show that the order in which 2 numbers are added or multiplied doesn't matter, the answer is always the same.

If a and b are real numbers, then

(A.3) $a + b = b + a$ (*addition*)

(A.4) $ab = ba$ (*multiplication*)

Example

changing the order

When two numbers are added or multiplied, there is more than one way of doing this:

adding 3 to 2	$\Rightarrow$	$2 + 3 = 5$
adding 2 to 3	$\Rightarrow$	$3 + 2 = 5$
multiplying 2 by 3	$\Rightarrow$	$2 \times 3 = 6$
multiplying 3 by 2	$\Rightarrow$	$3 \times 2 = 6$

The answer is the same in each case, and this is what the commutative rules for addition and multiplication describe.

We use the associative and commutative rules every time we rearrange sums and products of real numbers, for example when collecting like terms together or when simplifying products.

Example

collecting like terms

To simplify a sum like

$$2x + 3y + 4x + 5z + 7y,$$

we first collect like terms together:

$$2x + 3y + 4x + 5z + 7y = 2x + 4x + 3y + 7y + 5z = \text{ etc } \ldots$$

Example

rearranging powers

To simplify a product like

$$x^2y^3x^4z^5y^7,$$

we need to combine powers with the same base:

$$x^2y^3x^4z^5y = x^2x^4y^3y^7z^5 = \text{ etc } \ldots$$

Distributive Rules for Multiplication *over* Addition

The distributive rules describe how we expand brackets and how we factorise.

If a, b and c are real numbers, then

(A.5)	$a(b+c) = ab + ac$	(*left distribution*)
(A.6)	$(b+c)a = ba + ca$	(*right distribution*)

The distribution rules are also true when there are more than two terms inside the brackets. We use these rules every time we expand brackets and every time we simplify sums by combining like terms together.

expanding and simplifying

Example
To simplify a sum like

$$2(x+y) + 4(2x+z) + 7y,$$

we first expand brackets, then combine like terms:

$$2(x+y) + 4(2x+z) + 7y = 2x + 2y + 8x + 4z + 7y = 10x + 9y + 4z$$

... can you see where the left distributive rule (A.5) was used ... and the right distributive rule (A.6)?

expanding brackets

Example
To expand a product like

$$(x+2)(x+3),$$

we multiply out the brackets like this

$$(x+2)(x+3) = x(x+3) + 2(x+3) = x^2 + 3x + 2x + 6 = x^2 + 5x + 6$$

... can you see where the left distributive rule (A.6) was used ... and the two places where the right distributive rule (A.5) was used?

Rules for Subtraction and Division

In order to use algebra to manipulate expressions involving subtraction and division, we first need to talk about *negatives* and *reciprocals*.

The *negative* of a number a is the unique number (denoted by $-a$) such that

$$a + (-a) = (-a) + a = 0.$$

Now the difference between two numbers can be rewritten as the sum of the first and the negative of the second, and we can apply the rules of addition above to this sum.

If a and b are real numbers, then

$$a - b = a + (-b) = a + (-1)b. \tag{A.7}$$

This means that we can use the associative and commutative rules for addition when we rearrange sums and differences of real numbers, *provided that differences are interpreted as in (A.7).*

rearranging sums and differences

Example

To simplify an expression like

$$2x - 3y + 4x - 5z + 7y,$$

we write

$$2x - 3y + 4x - 5z + 7y = 2x + 4x - 3y + 7y - 5z = 6x + 4y - 5z$$

because we know

$$2x + (-3y) + 4x + (-5z) + 7y = 2x + 4x + (-3)y + 7y + (-5z) = 6x + 4y - 5z.$$

The *reciprocal* of a number a is the unique number (denoted a^{-1}) such that

$$a \times a^{-1} = a^{-1} \times a = 1.$$

Now the quotient of two numbers can be rewritten as a product of the numerator with the reciprocal of the denominator, so the rules for products above can be applied.

If a and $b \neq 0$ are real numbers, then

$$\frac{a}{b} = a \times b^{-1}$$

However, we need to be careful when dividing numbers, as the associative rule and the commutative rule are *not true* for division!

Example

$$(24 \div 6) \div 2 = 4 \div 2 = 2 \quad \text{and} \quad 24 \div (6 \div 2) = 24 \div 3 = 8.$$

$$6 \div 2 = 3 \quad \text{and} \quad 2 \div 6 = \frac{1}{3}$$

Appendix B

Answers

Exercise 1.1

1(a) pq (b) q (c) p   2(a) -2 (b) y (c) 0 (d) -3

3. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ 4. $B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ 5. $C = \begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix}$

Exercise 1.2

1. $\begin{bmatrix} 2 & -10 & 8 \\ 4 & 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 3/2 & 1/2 & 1 \\ 0 & 1/2 & 2 \end{bmatrix}$, $\begin{bmatrix} -3 & -13 & 8 \\ 6 & 1 & -8 \end{bmatrix}$

2. $a = 2$, $b = -2$, $c = -1$, and $d = \frac{1}{2}$ 3. $p = -\frac{1}{5}$, $q = \frac{4}{5}$ 4. $u = \frac{5}{7}$, $v = -\frac{8}{7}$

5. & 6. Check with a tutor.

Exercise 1.3

1(a) $5L - M$ (b) $10L$ 2. $X = -\begin{bmatrix} 2 & 4 \\ 4 & 0 \end{bmatrix}$ 3. $T = 4S$

4. Check with a tutor.

Exercise 1.4

1(a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$

(f) $\begin{bmatrix} -3 & -6 \\ 2 & 3 \end{bmatrix}$ (g) $\begin{bmatrix} -3 & 0 \\ -3 & 0 \end{bmatrix}$ (h) $\begin{bmatrix} -3 & 0 \\ -3 & 0 \end{bmatrix}$ (i) $\begin{bmatrix} 0 & -2 \\ 3 & -1 \end{bmatrix}$ (j) $\begin{bmatrix} -2 & -2 \\ 3 & -1 \end{bmatrix}$

2. 81

3(a) $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$; not possible   (b) $[-1]$; $\begin{bmatrix} 3 & -6 \\ 2 & -4 \end{bmatrix}$

Exercise 1.5

(a) $A^2 - AB + BA - B^2$ (b) $2AB + 2BA$

Exercise 1.7

3.$B^2 = 3B + 4I$, $B^3 = 13B + 12I$, $B^4 = 51B + 52I$

4. $A^9 = (A^3)^3 = (A)^3 = A \Longrightarrow A^{27} = (A^9)^3 = (A)^3 = A$
and $A^{31} = A^{27}A^3A = AAA = A^3 = A$

Exercise 1.8

1(a) [5] (b) $\begin{bmatrix} -1 & 3 \\ -2 & 6 \end{bmatrix}$

Exercise 2.1

(a) $\begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 4 & 1 \\ 2 & 0 & -1 \\ 4 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

Exercise 2.2

3(a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ (b) $X = \begin{bmatrix} -5 & 10 \\ 8 & -14 \end{bmatrix}$ (c) $X = \begin{bmatrix} -4 & -1 & 4 \\ -6 & -1 & 5 \end{bmatrix}$

4. $A^{-1} = 3I - A$ 5. & 6. Check with a tutor.

Exercise 2.3

1. $-\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$ 2 & 3. Check with a tutor. 4. 0; no

Exercise 2.4

(a) $\frac{1}{5}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$ (b) not possible

(c) $\frac{1}{4}\begin{bmatrix} 2 & 0 & -2 \\ -5 & 2 & 5 \\ -3 & 2 & -1 \end{bmatrix}$ (d) not possible