MathsTrack

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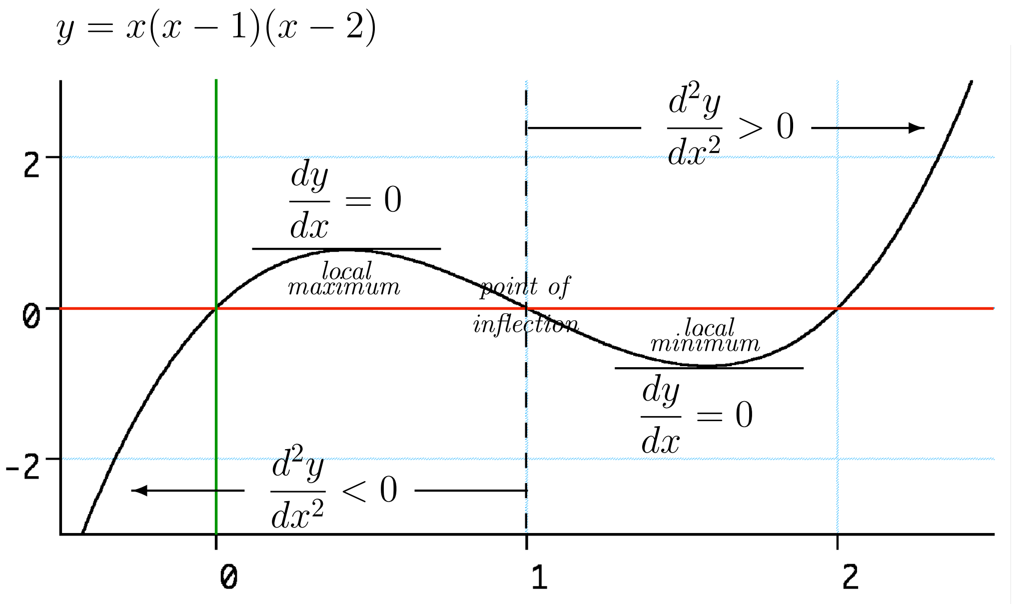
New books will be created during 2013 and 2014)

THE UNIVERSITY of ADELAIDE

Module 7

Applications of Differentiation



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This Topic . . .

This topic uses differentiation to explore

- the graphs of functions
- optimisation problems
- motion in a straight line

The topic has 2 chapters:

Chapter 1 explores functions and their graphs. It introduces continuity and then examines where functions are increasing or decreasing and have maximum or minimum values, where their rates of change increase or decrease and have maximum or minimum values, and the behaviour of rational functions near points where they are not defined and for large values of the variable. *A graphing calculator is not necessary for this module.*

Chapter 2 investigates optimisation problems. It introduces local maxima and minima of functions, and explores how to find these from mathematical models by differentiation, and also when constraints are present.

Auhor: Dr Paul Andrew Printed: February 24, 2013

Contents

Chapter 1

Functions and their graphs

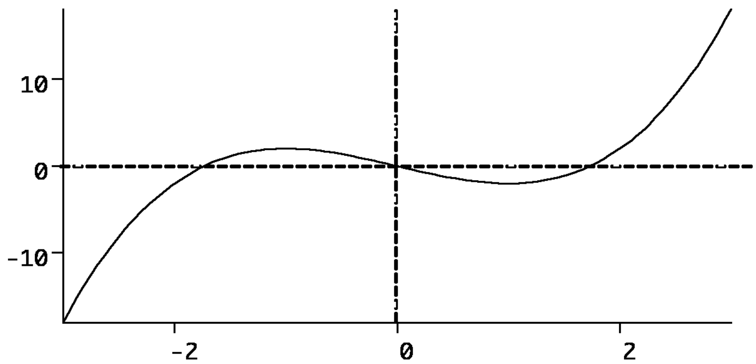
1.1 Introduction

How are functions described? What should we look for when investigating functions?

Example

graph of a function

This is a graph of the cubic function $y = x(x^2 - 3)$.



The graph shows how $x(x^2 - 3)$ changes when x changes.

Many features of functions correspond to features on their graphs. These are used when describing the *behaviour of a function*[1] and when interpreting mathematical models.

For example, on graphs of functions

- the x-intercepts correspond to the zeros of the function
- the y-intercept corresponds to the initial value $f(0)$ of the function, which is important in mathematical models

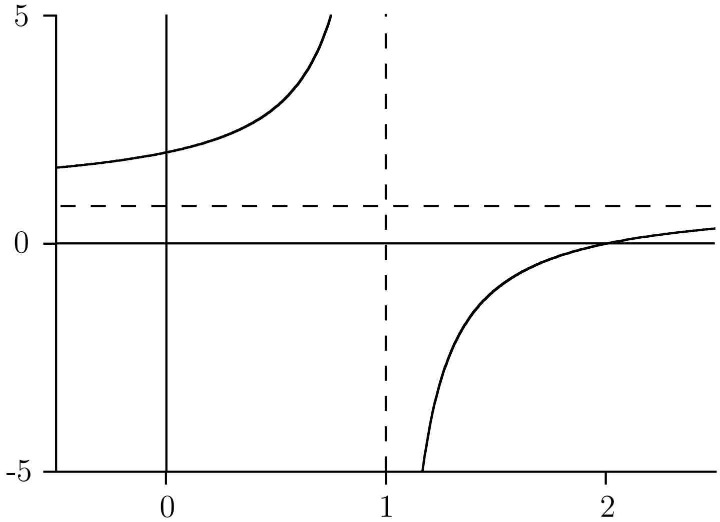
[1]Describing how a function $f(x)$ changes when x changes..

- the shape of the curve gives information on when and how quickly a function increases or decreases.

Example

vertical & horizontal asymptotes

This is a graph of the rational function $y = \dfrac{x-2}{x-1} = 1 - \dfrac{1}{x-1}$, $x \neq 1$.



The graph has vertical asymptote $x = 1$ and horizontal asymptote $y = 1$.

Vertical asymptotes correspond to zeros in the denominator of rational functions, where the function is not defined. The curve never cuts a vertical asymptote and its shape near the asymptote shows how a function behaves near the points where it is not defined.[2]

A horizontal asymptote shows the behaviour of a function as $x \to +\infty$ or $x \to -\infty$. The former corresponds to long-term behaviour in mathematical models.

[2]Asymptotes are revised in Appendix B.

Exercise 1.1

1. Sketch the graphs of the following functions, showing the x- and y-intercepts.[3]

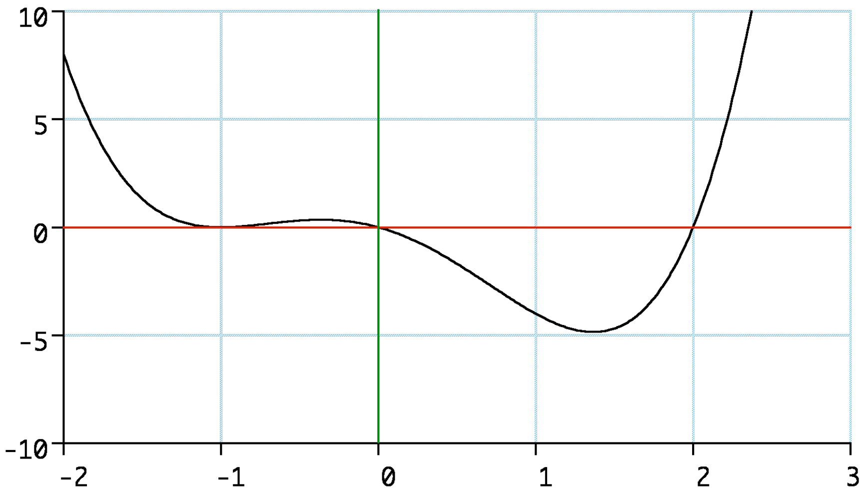
   (a) $y = x^2 - 3x + 2$

   (b) $y = (x-1)^2$

   (c) $y = (x-1)(x^2-4)$

   (d) $y = (x^2-1)^2$

2. What polynomial of degree 4 has graph:



3. What are the vertical and horizontal asymptotes of

   (a) $y = \dfrac{1}{x-1}, \quad x \neq 1$

   (b) $y = 1 + \dfrac{1}{x-1}, \quad x \neq 1$

   (c) $y = \dfrac{x-2}{x-1}, \quad x \neq 1$

---

[3]You may need to review Module 1 (Polynomials).

1.2 Continuity and the sign of a function

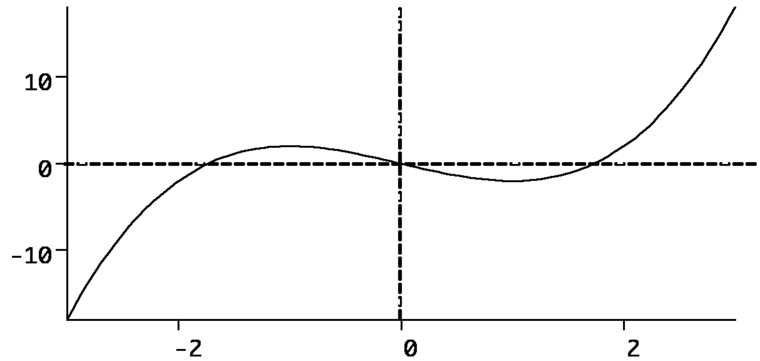
A function is said to be *continuous* on an open interval[4] if its graph is an unbroken curve[5] on the interval. It can be shown that:

- All polynomial functions are continuous on $(-\infty, \infty)$.
- A rational function is continuous on any open interval on which it is defined.[6,7]

Example

polynomial function

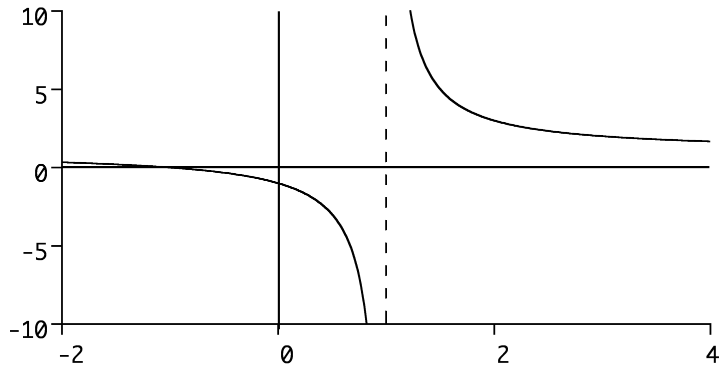
This is the graph of the cubic function $x(x^2 - 3)$. The function is continuous on $(-\infty, \infty)$.



Example

rational function

This is a graph of the rational function $y = \dfrac{x+1}{x-1}, x \neq 1$. The function is continuous on the intervals $(-\infty, 1)$ and $(1, \infty)$, and is not defined for $x = 1$.



The dotted line $x = 1$ is the vertical asymptote. Writing the function in the form

$$y = 1 + \frac{2}{x-1}$$

[4]Intervals are revised in Appendix A.

[5]In the sense that you can trace along the curve without taking your pencil off the paper.

[6]A rational function is the quotient of two polynomial functions.

[7]If two (or more) functions are continuous on an interval, then their sums, differences and products are also continuous on the interval.

shows that the horizontal asymptote is $y = 1$.

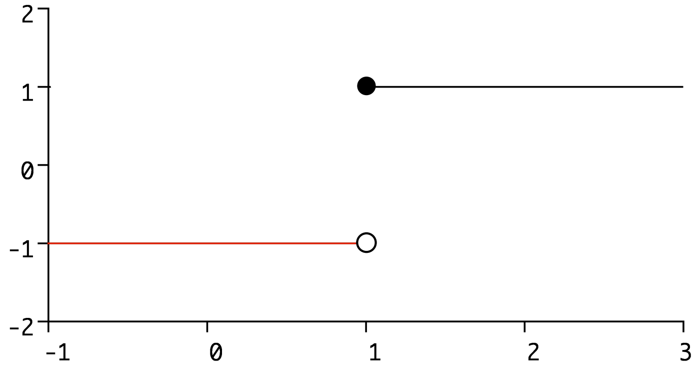
A function is said to be *discontinuous* if it is defined on an interval and if its graph is a curve on the interval that is broken into more than one piece.

Example

discontinuity at a point

This is the graph of the function

$$y = f(x) = \begin{cases} +1 & \text{if } x \geq 1 \\ -1 & \text{if } x < 1 \end{cases}$$



The function is defined on $(-\infty, \infty)$ but is discontinuous at $x = 1$. Notice, however, that it is continuous on both $(-\infty, 1)$ and $[1, \infty)$ separately.

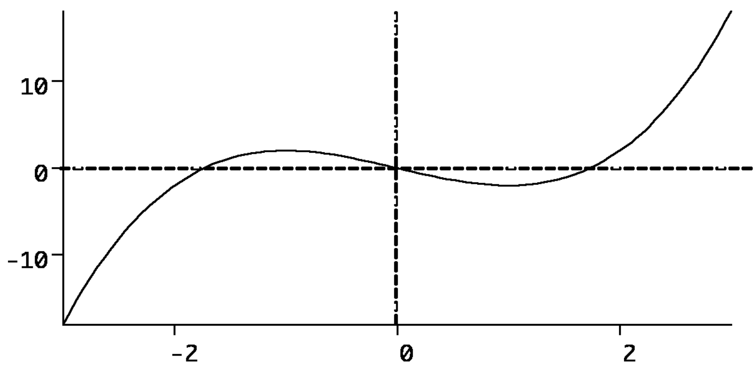
When a function is continuous we can describe its behaviour in general terms. One way of doing this is by using a sign diagram.

The value of a function is either positive or negative or zero. The corresponding signs of the function are $+$, $-$, and 0.

Example

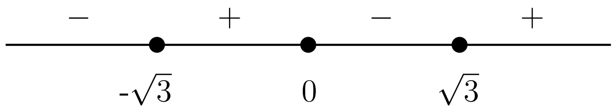
zeros intercepts signs

The cubic function $x(x^2 - 3)$ is continuous on $(-\infty, \infty)$ and has zeros at $-\sqrt{3}$, 0 and $\sqrt{3}$.



You can see from the graph that $x(x^2-3)$ is positive when $-\sqrt{3} < x < 0$ and $\sqrt{3} < x < \infty$, and is negative when $-\infty < x < -\sqrt{3}$ and $0 < x < \sqrt{3}$.

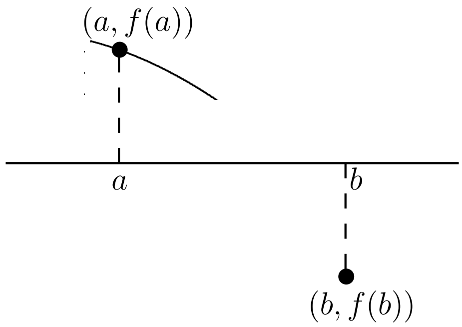
The sign diagram for $x(x^2-3)$ shows how the sign of the function depends upon x:



When a function is *continuous* on $(-\infty, \infty)$ it always has the same sign

- before its smallest zero
- between consecutive zeros
- after its largest zero

The diagram below shows why. If a function is positive at $x = a$ and negative at $x = b$ (or vice-versa) then its graph must cut the x-axis somewhere in (a, b).



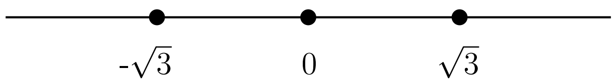
This observation enables us to draw a sign diagram of a function without using its graph:

1. Find the zeros of the function, then
2. select convenient points to test the sign of the function
   (i) before the smallest zero
   (ii) between consecutive zeros
   (iii) after the largest zero

Example

testing the sign

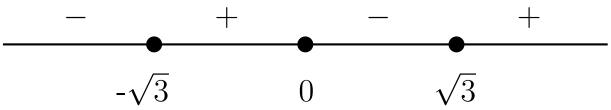
The zeros of $x(x^2-3)$ are $-\sqrt{3}, 0$ and $\sqrt{3}$.



Select convenient points to test the sign of $x(x^2-3)$:

$$\begin{array}{rcl} x=-2 & \Rightarrow & x(x^2-3)=-2<0 \\ x=-1 & \Rightarrow & x(x^2-3)=2>0 \\ x=1 & \Rightarrow & x(x^2-3)=-2<0 \\ x=2 & \Rightarrow & x(x^2-3)=2>0 \end{array}$$

The sign diagram for $x(x^2-3)$ is:

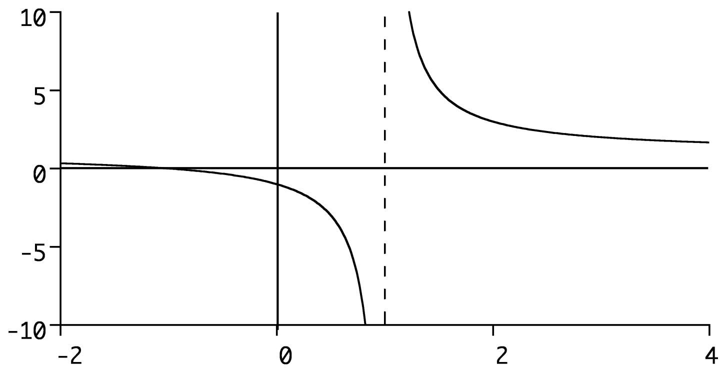


Sign diagrams need to be adjusted when functions are not defined or not continuous on all of $(-\infty.\infty)$.

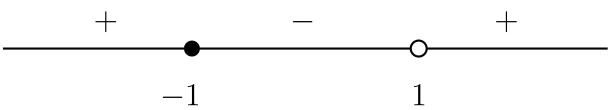
Example

missing value

The rational function $\dfrac{x+1}{x-1}, x \neq 1$ is not defined at $x=1$. It has a zero at $x=-1$ and is continuous on the intervals $(-\infty,1)$ and $(1,\infty)$.



As the function is continuous on $(-\infty,1)$ and $(1,\infty)$, we can represent the missing value $x=1$ by a hollow circle on the sign diagram and then test the sign of the function by selecting points between and on either side of -1 and 1.



Exercise 1.2

1. Draw sign diagrams for the following functions.

   (a) $y = x - 1$

   (b) $y = (x-1)(x-2)$

   (c) $y = (x-1)^2$

   (d) $y = (x-1)(x-2)(x-3)$

   (e) $y = (x-1)^2(x-2)$

   (f) $y = (x-1)^3$

   (g) $y = 1 + \dfrac{1}{x-1}$

   (h) $y = 1 - \dfrac{10}{(x-1)(x+2)}$

1.3 Increasing and decreasing functions

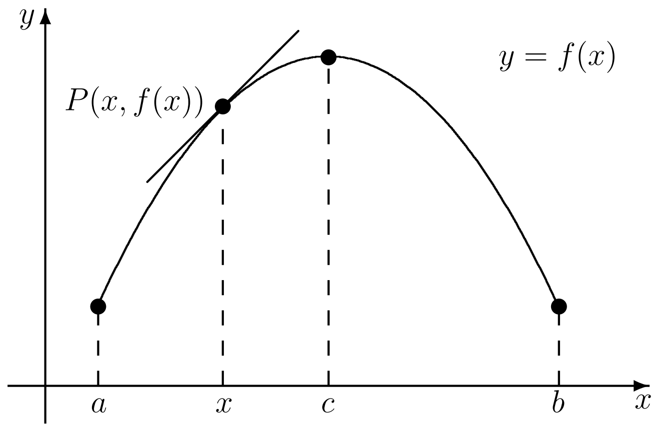
We can find further information about the behaviour of a function by looking at its derivative.

increasing function

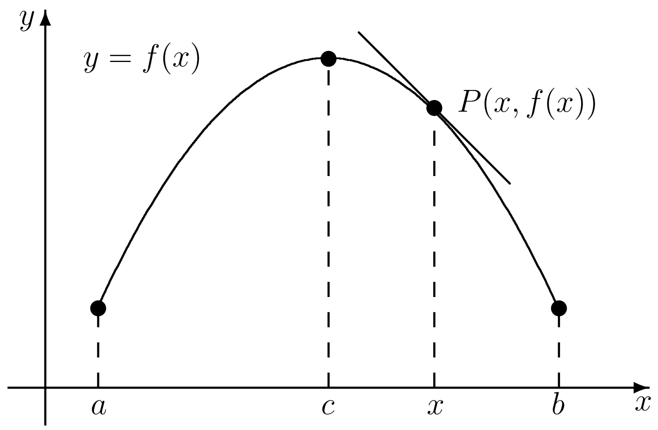
decreasing function

Example

Consider the graph of $f(x)$ below on the interval $[a, b]$. The gradient of the curve is positive for all points P between $x = a$ and $x = c$ because the tangent line has a positive gradient. You can also see that the value of the function increases as x increases from a to c. We say that $f(x)$ is an *increasing function* on the interval $[a, c]$.



If P is between $x = c$ and $x = b$ then the gradient of tangent line at P is negative. You can also see that the value of $f(x)$ decreases as x increases from c to b. We say that $f(x)$ is an *decreasing function* on the interval $[c, b]$.



Functions which are always increasing or always decreasing on an interval are called *monotonic functions*. For example, x^2 is monotonic on $x \geq 0$ and e^x is monotonic for all x.

> If the function $f(x)$ is defined on interval $[a, b]$ and $\dfrac{dy}{dx} > 0$ for all x in (a, b) , then $f(x)$ is an increasing function on $[a, b]$.
>
> If the function $f(x)$ is defined on interval $[c, b]$ and $\dfrac{dy}{dx} < 0$ for all x in (c, b), then $f(x)$ is a decreasing function on $[c, b]$.

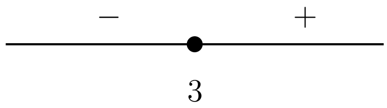
The intervals where a function is increasing or decreasing can be found by using the sign diagram of its derivative.

Example

sign diagram

The function $f(x) = x^2 - 6x + 8$ has derivative $f'(x) = 2x - 6$. The derivative is continous on $(-\infty, \infty)$ and has a zero at $x = 3$.

As $f'(2) < 0$ and $f'(4) > 0$, the sign diagram of $f'(x)$ is:



This shows that $f(x) = x^2 - 6x + 8$ is decreasing on $(-\infty, 3]$ and increasing on $[3, \infty)$.[8, 9]

Exercise 1.3

1. Draw sign diagrams for $x(x^2 - 3)$ and its derivative. Use these to find where the function is:
   (i) non-negative
   (ii) negative
   (iii) increasing
   (iv) decreasing
2. Repeat question 1 for the function $x^2(x - 3)$.
3. Repeat question 1 for the function $y = 1 + \dfrac{1}{x-1}, x \neq 1$

[8]The graph of $y = f(x) = x^2 - 6x + 8$ is a parabola with vertex at $x = 3$.
[9]The description *increasing/decreasing* can be extended to open or half-open intervals.

1.4 Stationary points

A point on the graph of $y = f(x)$ that has a horizontal tangent line is called a *stationary point*. Stationary points correspond to zeros of the derivative.

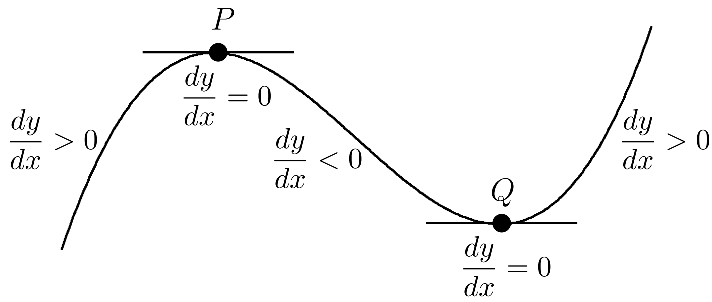
There are two types of stationary points:

- turning points
- horizontal points of inflection

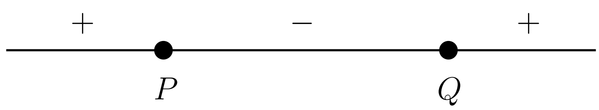
Example

turning points

The points P and Q on the graph of $y = f(x)$ below are stationary points as their tangent lines are horizontal. P and Q are also called *turning points* because the graph 'turns around' at each point. You can see that as x increases, the curve stops increasing at P and begins to decrease, then stops decreasing at Q and begins to increase.



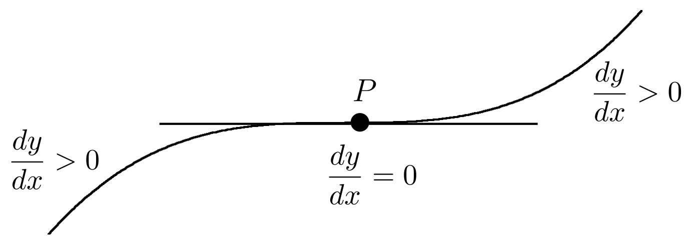
The sign diagram for the derivative shows how the derivative changes sign at each turning point.



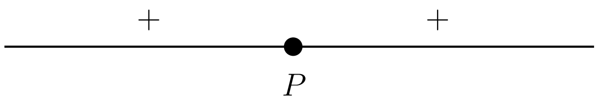
Example

point of inflection

The point P on the graph of $y = f(x)$ below is a stationary point. You can see that P is not a turning point as $f(x)$ is increasing on both sides of P. P is called a *horizontal point of inflection* because the graph 'bends' at P without turning around.



The sign diagram for the derivative shows that it has the same sign on both sides of the point of inflection.



Example

turning
inflection

(a) Find the stationary points on $y = x(x-1)^2$.

(b) Use the sign diagram for the derivative to decide whether they are turning points or horizontal points of inflection.

Answer

(a) The derivative of $y = x(x-1)^2$ is

$$\begin{aligned} \frac{dy}{dx} &= 1 \times (x-1)^2 + x \times 2(x-1) \\ &= (x-1)\{(x-1) + 2x\} \\ &= (x-1)(3x-1) \end{aligned}$$

The derivative has zeros at $x = 1/3$ and $x = 1$.

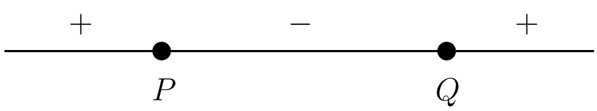
$$\begin{array}{lll} x = 1 & \Longrightarrow & y = 1 \times (1-1)^2 = 0 \\ x = 1/3 & \Longrightarrow & y = \frac{1}{3}(\frac{1}{3} - 1)^2 = 4/27 < 0 \end{array}$$

The stationary points are $(1/3, 4/27)$ and $(1, 0)$.

(b) Testing the sign of the derivative:

$$\begin{array}{llll} x = 0 & \Longrightarrow & (x-1)(3x-1) = (0-1)(0-1) > 0 & (+) \\ x = 1/2 & \Longrightarrow & (x-1)(3x-1) = (1/2-1)(3/2-1) < 0 & (-) \\ x = 2 & \Longrightarrow & (x-1)(3x-1) = (2-1)(6-1) > 0 & (+) \end{array}$$

The sign diagram for the derivative is:

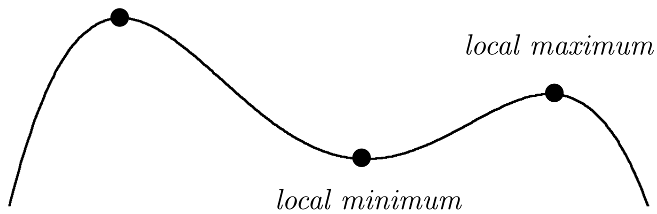


The points are both turning points.

The *global maximum* (*global minimum*) of a function is the greatest (least) value it takes in its domain.[10]

A function is said to have a *local maximum* (*local minimum*) at a turning point when it has the greatest (least) value in an interval containing the turning point and when this value is not its global maximum (global minimum).

[10]The domain of a function is the set of values for which it is defined.



Exercise 1.4

1. Find the stationary points (if any) of the curves below. Use a sign diagram to determine whether they are turning points or points of inflection.

   (a) $y = x^3 - 4x$

   (b) $y = x^3 - 9x^2 + 15x + 4$

   (c) $y = x^3 + 2x - 1$

   (d) $y = x^4 - 18x^2$

   (e) $y = 1 + \dfrac{1}{x-1}, x \neq 1$

   (f) $y = x + \dfrac{1}{x-1}, x \neq 1$

2. Classify all stationary points on the curves above as:

   - global/local maxima
   - global/local minima
   - horizontal point of inflection

3. The cubic $y = x^3 - ax, a > 0$ has a stationary point at $x = 1$.

   (a) Find the value of a.

   (b) Find the positions of all other stationary points.

   (c) Sketch the graph of $y = x^3 - ax$.

4. The parabola $y = ax^2 + bx + c, a \neq 0$ has a single stationary point. What is its x-coordinate? What condition determines whether it is a local maximum or a local minimum?

5. The function $f(x) = x^3 + ax + b$ has a stationary point at (1, 4). Find the values of a and b, and the position of all other stationary points.

6. The cubic curve $y = ax^3 + bx^2 + cx + d$ has a stationary point at $(1, 0)$ and touches the line $y = -9x + 5$ at $(0, 5)$. Find a, b, c and d.

1.5 The shape of a curve

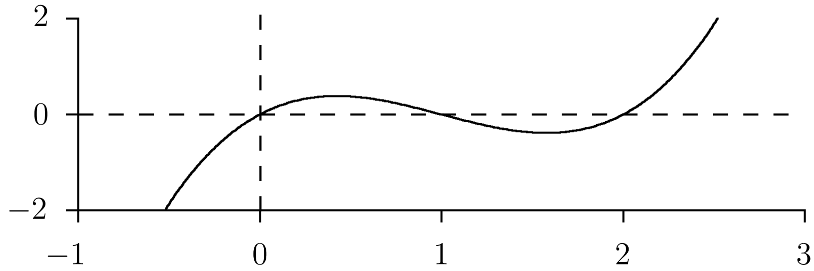
The shape of a curve is said to be *concave up* on an intervals when its gradient is increasing, and *concave down* when its gradient is decreasing. *The tangent line is always below a curve when it is concave up and above a curve when it is concave down.*[11]

A point where a curve changes shape from being concave up to concave down (or vice versa) is called a *point of inflection*.

Example

shape inflection

The graph of $y = x(x-1)(x-2)$ below is concave down on $(-\infty, 1]$ and concave up on $[1, \infty)$. The point of inflection is $(1, 0)$.



We need to investigate *the derivative of the derivative* of a function $f(x)$

$$\frac{d}{dx}(\frac{dy}{dx})$$

to determine whether its derivative is increasing (concave up) or decreasing (concave down). This is called the *second derivative* of the function and is commonly denoted by the symbols

$$\frac{d^2y}{dx^2} \quad \text{(pronounced "dee squared y dee x squared")}$$

or

$$y'' \quad \text{(pronounced as "y double dash")},$$

or alternatively, $\frac{d^2f}{dx^2}$ and $f''(x)$.

The graph of $y = f(x)$ is

- concave up on $[a, b]$ when $y'' > 0$ for all x in (a, b)
- concave down on $[a, b]$ when $y'' < 0$ for all x in (a, b)

[11]Some texts use the terms *convex* and *concave* instead of *concave up* and *concave down*, but we will not be using the term "concave" in this text.

A point of inflection is where a curve changes shape from being concave up to concave down (or vice versa). In other words the second derivative y'' must change signs.

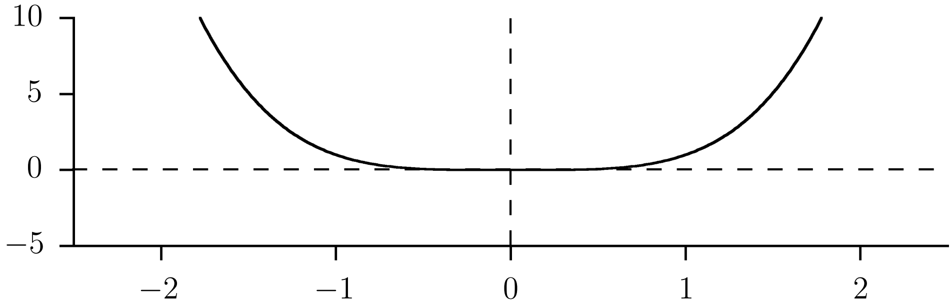
The graph of $y = f(x)$ has a point of inflection at P if

- $y'' = 0$ at P
- y'' changes sign at P

Example

sign and inflection

The graph of $y = x^4$ below has $y'' = 0$ at $(0, 0)$. This is not a point of inflection as the gradient is increasing on both sides of $(0, 0)$, that is $y'' = 12x^2 > 0$ on each side of $(0, 0)$.



Example

sign and inflection

Where is the graph of $y = x(x - 1)(x - 2)$ concave up and concave down? What is the point of inflection?

Answer

The derivative of $y = x(x - 1)(x - 2) = x^3 - 3x^2 + 2x$ is

$$\frac{dy}{dx} = 3x^2 - 6x + 2$$

The second derivative is

$$\frac{d^2y}{dx^2} = 6x - 6$$

This is zero at $x = 1$.

As

$$\frac{d^2y}{dx^2} = 6x - 6 \quad < \quad 0 \text{ when } x < 1 \text{ and}$$
$$\frac{d^2y}{dx^2} = 6x - 6 \quad > \quad 0 \text{ when } x > 1,$$

the curve is concave down on $(-\infty, 1]$ and concave up on $[1, \infty)$, and $(1, 0)$ is a point of inflection.

Exercise 1.5

1. For each of the curves below

   - find the point(s) of inflection (if any)
   - describe where the curve is concave up and concave down.

   (a) $y = x^3 - 4x$

   (b) $y = x^3 - 9x^2 + 15x + 4$

   (c) $y = 1 - 2x - x^3$

   (d) $y = x^4 - 9x^2$

   (e) $y = 1 + \dfrac{1}{x-1}$

   (f) $y = x + \dfrac{1}{x-1}$

2. If $y = ax^3 + bx^2 + cx + d$ has two distinct turning points at $x = p$ and $x = q$, show that

   (i) $b^2 > 3ac$

   (ii) the point of inflection is at $x = \dfrac{p+q}{2}$.

1.6 Overview

The main features on the graph of $y = f(x)$ are:

- *continuity* and *asymptotes*
- *intercepts* on the x-axis ($y = 0$) and the y-axis ($x = 0$)
- the *sign* of the function ($+$, 0, $-$)
- intervals where the function is *increasing* ($y' > 0$) or *decreasing* ($y' < 0$)
- *stationary points* ($y' = 0$) and *turning points* ($y' = 0$, y' changes sign)
- the *shape* of the curve, *concave up* ($y'' > 0$) or *concave down* ($y'' < 0$)
- *points of inflection* ($y'' = 0$, y'' changes sign)

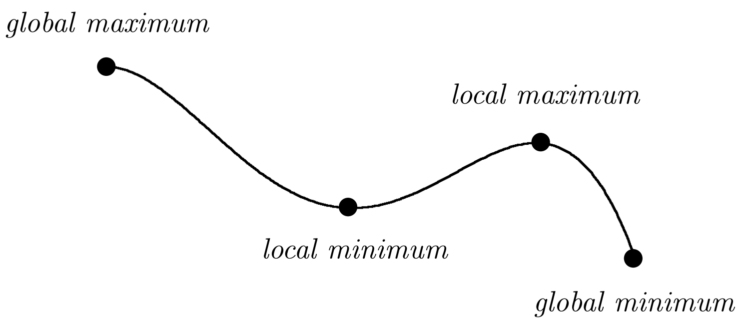
Chapter 2

Optimisation

2.1 Introduction

There are many problems where differentiation can be used to find the maximum or minimum value of a continuous function.

The global maximum and minimum values of a function do not always occur at stationary points. They may also occur at the endpoints of the domain of a continuous function.



stationary point

Example

The cost of manufacturing a batch of items is given by

$$C(x) = 4x^3 - 4800x + 120 \text{ dollars/item}$$

where x (≥ 1) is number of items manufactured in a batch. Find the size of a batch for which the cost per item is least.

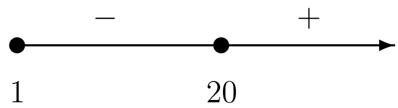
Answer

The stationary points of $C(x) = 4x^3 - 4800x + 120$ occur when the derivative $C'(x) = 12x^2 - 4800$ is zero.

$$\begin{aligned} 12x^2 - 4800 &= 0 \\ 12x^2 &= 4800 \\ x^2 &= 400 \\ x &= \pm 20 \end{aligned}$$

The negative solution is not feasible as $x \geq 1$.

The sign diagram for $C'(x)$ shows that the global minimum of $C(x)$ occurs at stationary point $x = 20$.



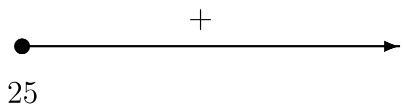
The least cost per item occurs in batches of 20 items

Example

global minimum

If the example above is changed so that only batches of size 25 or more are considered, then

1. the cost function $C(x)$ will now have domain $x \geq 25$
2. there will be no stationary points in the domain
3. the sign diagram of the derivative will become:



Here the sign diagram for $C'(x)$ shows that the global minimum of $C(x)$ occurs at endpoint $x = 25$.

An alternate way of deciding if a stationary point is a local maximum or a local minimum is to consider the second derivative of a function. See page 16.

2.2 Maxima-Minima

The following examples show how to construct and analyse functions in order to solve an optimisation problem.[1]

To solve an optimisation problem using differentiation:

1. Express the quantity to be optimised as a function of one variable.
2. Find the derivative with respect to the variable, then use it to find any stationary points.
3. Examine the sign of the derivative to see if the stationary points are global maxima/minima, taking account of physical constraints.

Example

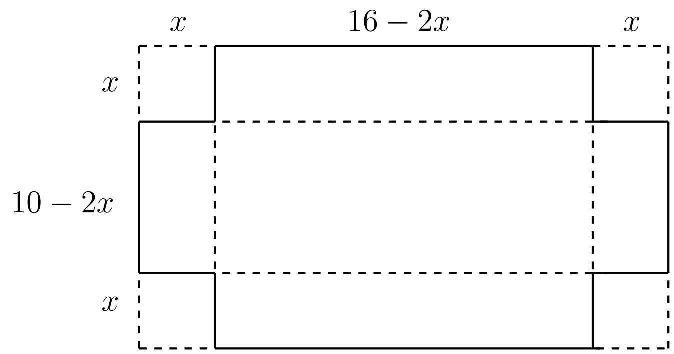
physical constraints

An open rectangular box is made by cutting out equal sized squares from each corner of a rectangular piece of cardboard having width 10 cm and length 16 cm.

What sized squares must be cut out to produce a box with maximum volume?

Answer

Let the squares have sides of length x cm. The diagram shows that $0 < x < 5$.



The volume of the box is:

$$V = x(10-2x)(16-2x)$$

Differentiating (product rule):

$$\begin{aligned}\frac{dV}{dx} &= \frac{d}{dx}(10x-2x^2)\times(16-2x)+(10x-2x^2)\times\frac{d}{dx}(16-2x)\\ &= (10-4x)(16-2x)+(10x-2x^2)(-2)\\ &= 12x^2-104x+160\end{aligned}$$

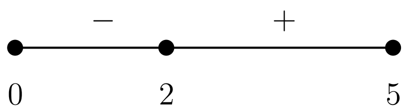
[1]To optimise means to find the maximum or the minimum.

The stationary points are found by solving $\dfrac{dV}{dx} = 0$:[2]

$$\begin{aligned} 12x^2 - 104x + 160 &= 0 \\ 3x^2 - 26x + 40 &= 0 \\ (3x-20)(x-2) &= 0 \\ x &= \begin{cases} 2 \\ 20/3 \end{cases} \end{aligned}$$

We know that $0 < x < 5$ by the physical constraints of the problem, so $x = 2$ is the only possible solution.

The sign diagram for $\dfrac{dV}{dx}$ shows that the global minimum of $V(x)$ occurs at $x = 2$.



The maximum volume is obtained by cutting out squares with 2 cm sides. It is:

$$V = x(10-2x)(16-2x) = 2 \times 6 \times 12 = 144 \text{ cm}^3.$$

Note: It can be useful to draw one or more diagrams when solving problems.

Also, it is can be easier to initially represent a situation by more than one equation, and then combine these to obtain the function of one variable that is be optimised.

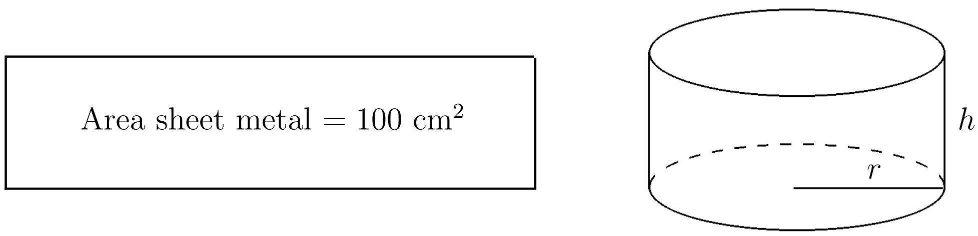
Example

combining equations

A manufacturer wishes to make an cylinder that is open at one end from 100 cm^2 of sheet metal. What radius would give the maximum volume?

Answer

Let the open cylinder have radius r cm and height h cm.



[2]This equation can also be solved by using the quadratic formula:

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ when } b^2 - 4ac \geq 0.$$

.

The surface area of the open cylinder is equal to the area of the metal sheet:

$$\pi r^2 + 2\pi r h = 100$$

The volume to be optimised is:

$$V = \pi r^2 h$$

We first need to express V in terms of a one variable ...say r.

To do this we need to rearrange the first formula so that h is the subject and then use this to replace the h in the second formula.

$$\begin{aligned} \pi r^2 + 2\pi r h &= 100 \\ 2\pi r h &= 100 - \pi r^2 \\ h &= \frac{100 - \pi r^2}{2\pi r} \quad (*) \end{aligned}$$

Notice (*) *shows that, for h to be positive, r is constrained to ...*

$$0 < r < \sqrt{\frac{100}{\pi}}.$$

Writing the volume as a function of r :

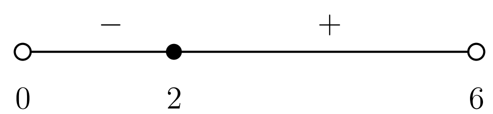
$$\begin{aligned} V &= \pi r^2 h \\ &= \pi r^2 h \\ &= \pi r^2 \times \frac{100 - \pi r^2}{2\pi r} \\ &= 50r - \frac{\pi}{2} r^3 \end{aligned}$$

The stationary points are found by solving $\frac{dV}{dr} = 0$:

$$\begin{aligned} 50 - \frac{3\pi}{2} r^2 &= 0 \\ r^2 &= \frac{100}{3\pi} \\ r &= \pm\sqrt{\frac{100}{3\pi}} \\ r &= \pm\frac{10}{\sqrt{3\pi}} \end{aligned}$$

We know that $0 < r < \sqrt{100/\pi}$ by the physical constraints of the problem, so $r = \frac{10}{\sqrt{3\pi}}$ is the only possible solution.

The sign diagram for $\frac{dV}{dr}$ shows that the global maximum of $V(r)$ occurs at $r = \frac{10}{\sqrt{3\pi}} \approx 3.26$ cm.

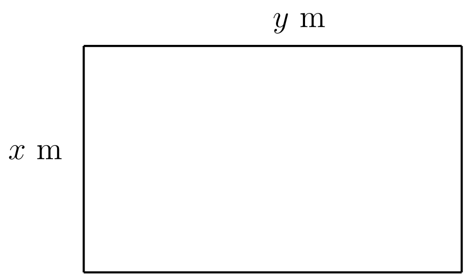


The corresponding volume is:

$$\begin{aligned} V &= 50r - \frac{\pi}{2}r^3 \\ &= 50 \times \frac{10}{\sqrt{3\pi}} - \frac{\pi}{2} \times \left(\frac{10}{\sqrt{3\pi}}\right)^3 \\ &= \frac{1000}{3\sqrt{3\pi}} \\ &\approx 108.58 \text{ cm}^3 \end{aligned}$$

Exercise 2.2

1. A farmer has 400 m of spare fencing and wants to build a rectangular garden enclosed by a fence as in the diagram below.



   (a) Express y in terms of x.

   (b) Show that the area, A m^2, of the garden is modelled by the function

   $$A = x(200 - x) \text{ m}^2$$

   for $0 < x < 200$.

   (c) Find $\frac{dA}{dx}$ and the stationary points of A.

   (d) Show that the area is a maximum when $x = 100$, and calculate this area.

2. A farmer decides to build a rectangular enclosure with area 200 m^2 using an existing fence as one side and with new fencing on the other three new sides.

   (a) If the sides meeting the existing fence each have length x m, show that the length, L m, of the new fencing is modelled by the function

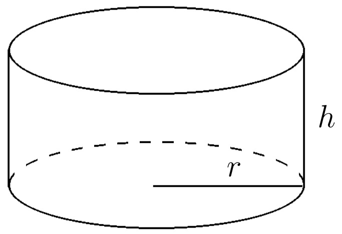
   $$L = 2x + \frac{200}{x}$$

   for $x > 0$.

   (b) Find $\frac{dL}{dx}$ and the stationary points of L.

(c) Find the value of x for which L is a minimum and calculate the minimum amount of new fencing needed.

3. A manufacturer wants to produce one litre cans from as small a quantity of metal sheeting as possible to reduce costs.



(a) Show that the height h cm of a one litre can with radius r cm is given by

$$h = \frac{1000}{\pi r^2}$$

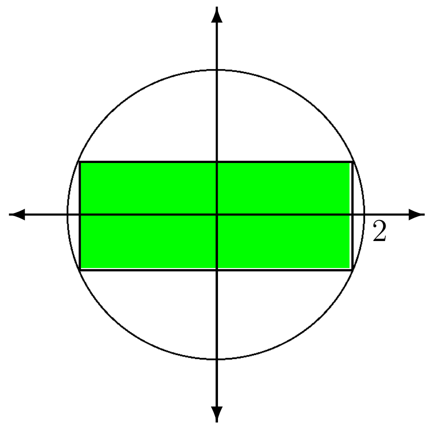
for $r > 0$.

(b) Show that the total surface area, A cm^2, of each can is modelled by the function

$$A = 2\pi r^2 + \frac{2000}{r}.$$

(c) Find the value of r for which A is a minimum, and calculate the minimum surface area of a one litre can.

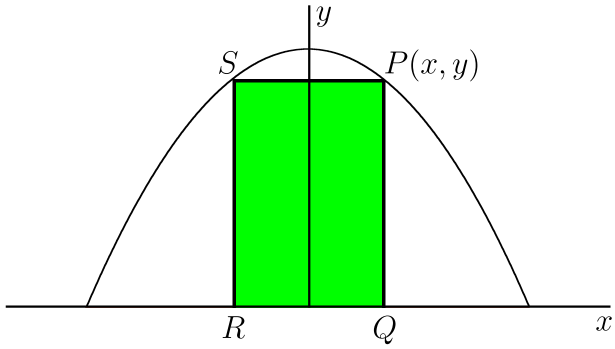
4. The diagram below shows a rectangle inscribed[3] in a circle of radius 2 cm.



Construct a model for the area of an inscribed rectangle, and use it to find the dimensions of the inscribed rectangle with greatest area.

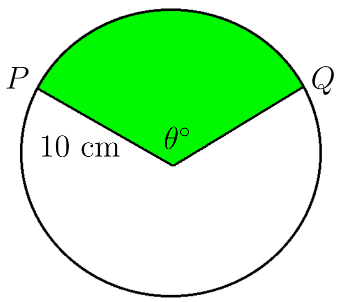
[3]Each corner of an inscribed rectangle lies on a circle. There are infinitely many inscribed rectangles for any circle.

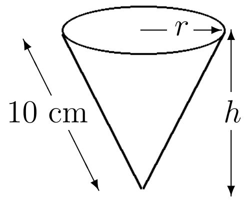
5. The graph of $y = 3 - x^2$ is shown below for $-\sqrt{3} \leq x \leq \sqrt{3}$. The rectangle $PQRS$ has base on the x-axis, is symmetric about the y-axis, and has corners P and S on the parabola.



Find the co-ordinates of $P(x, y)$ for which the area of $PQRS$ is greatest.

6. A sector with angle $\theta°$ and radius 10 cm is bent to form a cone with radius r cm and height h cm as in the diagram below.





(a) Show that

- arc PQ has length $\dfrac{\theta\pi}{18}$ cm.
- the radius of the cone is $r = \dfrac{\theta}{36}$.
- the height of the cone is $h = \sqrt{100 - \left(\dfrac{\theta}{36}\right)^2}$, for $0 \leq \theta \leq 360°$

(b) Construct a function of θ that models the volume, V cm^3, of the cone.

(c) Find $\dfrac{dV}{d\theta}$ and the stationary points of $V(\theta)$.

(d) Find the value of θ for which V is a maximum and calculate the greatest volume of the cone.

Appendix A

Intervals on the real line

The most common subsets of the real line are intervals. An *interval* is a segment of the real line between two *endpoints*.

Example

interval
endpoint

The interval drawn on the x-axis below represents the set of x-values between endpoints 1 and 3.

The filled circle at 1 indicates that the interval is *closed* at 1 (that 1 is included in the interval), and the empty circle at 3 indicates that the interval is *open* at 3 (that 3 is not included in the interval).

This interval can be represented using *interval notation* as $[1, 3)$. Here

- the left square bracket $[\ldots$ indicates that the left endpoint is included in the interval.
- the right round bracket $\ldots)$ indicates that the right endpoint is not included in the interval.

The interval $[1, 3)$ is read as

the interval from 1 to 3, including 1 and excluding 3'.

Intervals that are open at both ends are called *open intervals*, and those that are closed at both ends are called *closed intervals*.

We refer to x-values in an interval by using the symbol $\in$, which means *belonging to* or *in*.

Example

belongs to
in

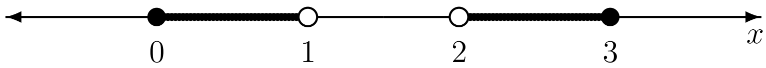
The function $f(x) = \sqrt{1 - x^2}$ is defined for all $x \in [-1, 1]$. This last part of the sentence is read aloud as "x is in the interval from -1 (including -1) to 1 (including 1)".

We combine two or more intervals by using the symbol $\cup$ (pronounced *union*).

Example

combining intervals

The union of $[0, 1)$ and $(2, 3]$ is $[0, 1) \cup (2, 3]$.



Some special subsets of the real line are:

(a) the real line itself. This doesn't contain any endpoints and can be represented by the open interval $(-\infty, \infty)$ or the special symbol $\mathbb{R}$.

(b) the set of positive numbers. This is represented by either the open interval $(0, \infty)$ or the symbol $\mathbb{R}^+$ or the inequality $x > 0$.

(c) the set of all non-negative numbers. This is represented by the half-open interval $[0, \infty)$ or the inequality $x \geq 0$.

Example

special intervals

The symbol $\mathbb{R}$ is often used when describing the domain of a function.

1. The function $\sqrt{1 + x^2}$ is defined for $x \in \mathbb{R}$.
2. The function $\dfrac{1}{\sqrt{x}}$ has domain $\mathbb{R}^+$.
3. The function $\sqrt{x}$ is defined for $x \geq 0$.

Intervals can also be described using *set notation.*

Example

set notation

The interval $[1, 3)$ can be written in set notation as $\{x : 1 \leq x < 3\}$. Here

- the curly brackets $\{ \ldots \}$ indicate a set or collection of numbers
- x is the variable taking the values on the real line
- the colon ' : ' stands for *such that* or *for which* (some people use the $|$ symbol instead of a colon.)
- the inequality shows the actual values that x takes.

The set $\{x : 1 \leq x < 3\}$ is read as

the set of x for which x is greater than or equal to 1 and less than 3.

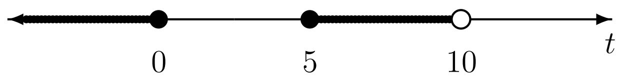
Exercise A

1. Represent the following intervals using (i) interval notation and (ii) set notation.

   (a) 

   (b) 

   (c) 

   (d) 

2. Draw the following intervals.

   (a) $[-1, 1]$
   (b) $[0, 5)$
   (c) $(1, \infty)$
   (d) $\mathbb{R}^-$
   (e) $\{x : 0 \leq x \leq 2\}$
   (f) $\{w : -1 < w < 1\}$

Appendix B

Asymptotes of rational functions

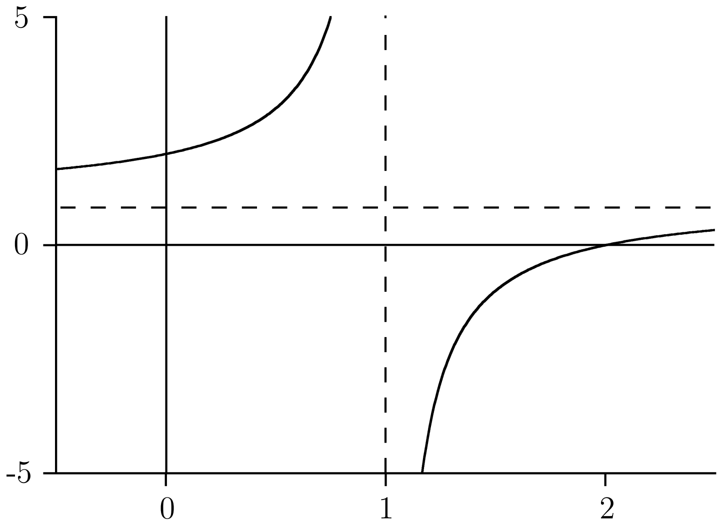
B.1 Vertical asymptotes

A rational function becomes large without bound (either positive or negative) as it takes values increasingly close to a zero in its denominator. The vertical line(s) through these zeros are called vertical asymptote(s).

Example

vertical & horizontal asymptotes

The graph of $y = \dfrac{x-2}{x-1} = 1 - \dfrac{1}{x-1}$, $x \neq 1$ is shown below. The vertical and horizontal asymptotes are indicated by the dotted lines $x = 1$ and $y = 1$.



You can see that the value of $y = \dfrac{x-2}{x-1}$ is very large when x is near 1.

x	0.9	0.99	0.999	0.9999
y	11	111	1111	11111

x	1.1	1.01	1.001	1.0001
y	-9	-99	-999	-9999

The graph and tables show that the behaviour of $y = \dfrac{x-2}{x-1}$ near $x = 1$ needs to be described separately for $x < 1$ and for $x > 1$. In particular:

1. As x approaches (becomes close to) 1 *from below*, y tends to $+\infty$ (becomes positive large without bound). This is written in symbols as[1, 2]

$$x \to 1^- \Rightarrow y \to +\infty$$

2. As x approaches (becomes close to) 1 *from above*, y tends to $-\infty$ (becomes negative large without bound). This is written in symbols as[3]

$$x \to 1^+ \Rightarrow y \to -\infty$$

The behaviour of a rational function near a zero in its denominator can be found directly from the function without using a graph or table:

1. when $x - 1$ is very small *and negative*, then[4]

$$y = \frac{x-2}{x-1} \approx \frac{1-2}{x-1} = \frac{-1}{x-1}$$

is very large *and positive*, so $x \to 1^- \Rightarrow y \to +\infty$.

2. when $x - 1$ is very small *and positive*, then

$$y = \frac{x-2}{x-1} \approx \frac{1-2}{x-1} = \frac{-1}{x-1}$$

is very large *and negative*, so $x \to 1^+ \Rightarrow y \to -\infty$.

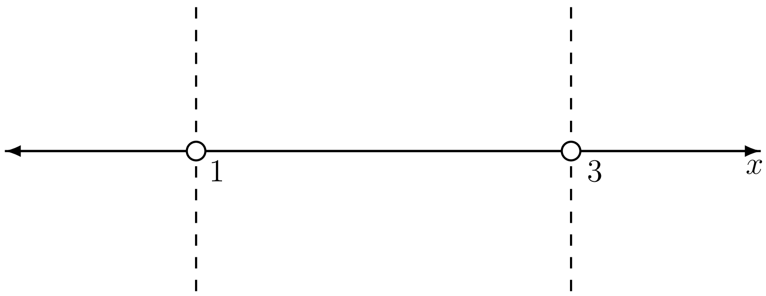
Example

interval endpoint

Sketch the graph of $y = \dfrac{10}{(x-1)(x-3)}, x \neq 1, 3$ near $x = 1$ and $x = 3$.

Answer

The denominator has zeros 1 and 3, so the vertical asymptotes are the lines $x = 1$ and $x = 3$.



[1] $x \to 1^-$ means *as x approaches 1 through values which are less than 1.*

[2] $\Rightarrow$ means *implies.*

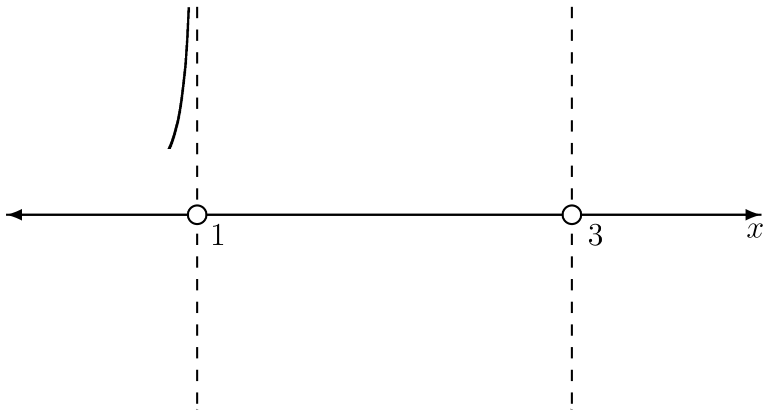
[3] $x \to 1^+$ means *as x approaches 1 through values which are greater than 1.*

[4] The symbol $\approx$ means *is approximately equal to.*

(1) When $x - 1$ is very small *and negative*, then

$$y = \frac{10}{(x-1)(x-3)} \approx \frac{10}{(x-1)(1-3)} = \frac{-5}{x-1}$$

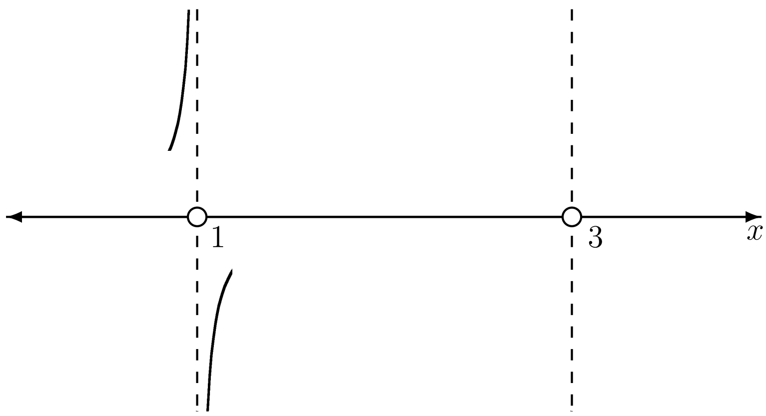
is very large *and positive*, so $x \to 1^- \Rightarrow y \to +\infty$.



(2) When $x - 1$ is very small *and positive*, then

$$y = \frac{10}{(x-1)(x-3)} \approx \frac{10}{(x-1)(1-3)} = \frac{-5}{x-1}$$

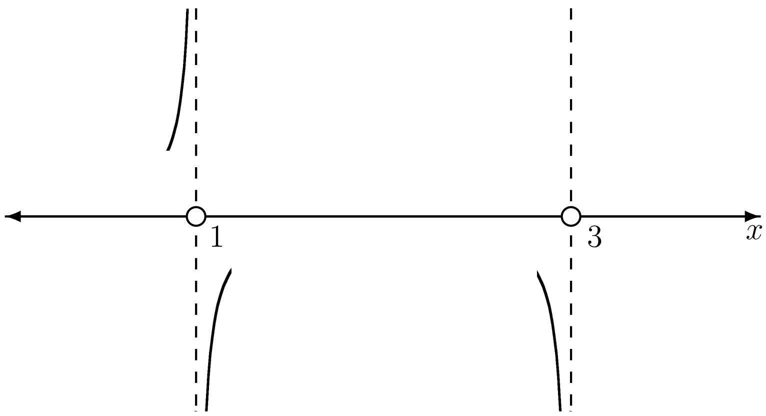
is very large *and negative*, so $x \to 1^+ \Rightarrow y \to -\infty$.



(3) When $x - 3$ is very small *and negative*, then

$$y = \frac{10}{(x-1)(x-3)} \approx \frac{10}{(3-1)(x-3)} = \frac{5}{x-3}$$

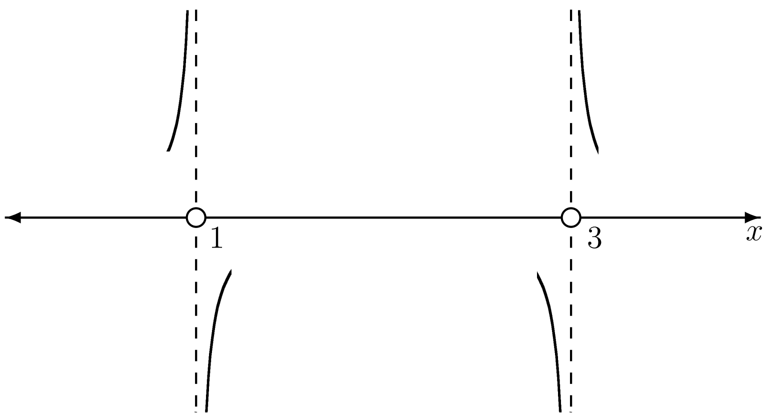
is very large *and negative*, so $x \to 3^- \Rightarrow y \to -\infty$.



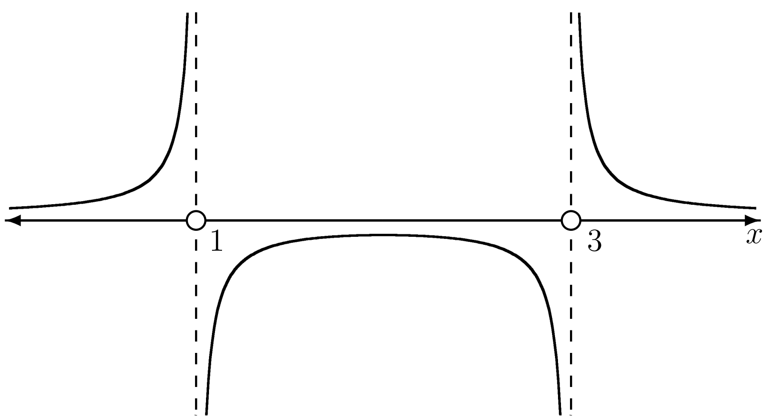
(4) When $x - 3$ is very small *and positive*, then

$$y = \frac{10}{(x-1)(x-3)} \approx \frac{10}{(3-1)(x-3)} = \frac{5}{x-3}$$

is very large *and positive*, so $x \to 3^+ \Rightarrow y \to +\infty$.



The actual graph of $y = \dfrac{10}{(x-1)(x-3)}, x \neq 1, 3$ is ...



Exercise B.1

1. What are the vertical asymptotes of
   (a) $y = \dfrac{x}{x+1}, x \neq -1$
   (b) $y = \dfrac{1}{1-x^2}, x \neq \pm 1$
   (c) $y = 1 + \dfrac{1}{x-2}, x \neq 2$
   (d) $y = \dfrac{1}{x-1} + \dfrac{1}{x+1}, x \neq \pm 1$
2. Sketch each of the graphs above near their vertical asymptotes.

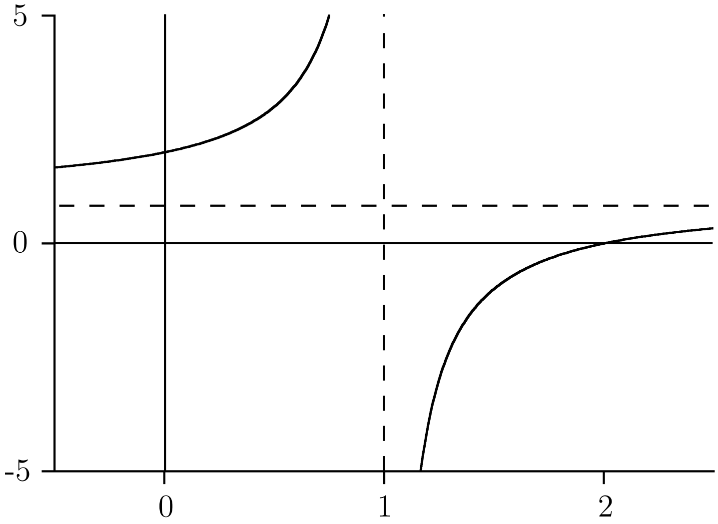
B.2 Horizontal asymptotes

If a rational function $f(x)$ approaches a constant c as $x \to \pm\infty$, then the line $y = c$ is called a horizontal asymptote of $y = f(x)$. The curve is approximated by the line $y = c$ when x is large.

Example

horizontal asymptote

The graph of $y = \dfrac{x-2}{x-1} = 1 - \dfrac{1}{x-1}$, $x \neq 1$ has horizontal asymptote $y = 1$.



The value of $y = \dfrac{x-2}{x-1}$ becomes close to 1 when x becomes very large (positive or negative).

x	11	101	1001	10001
y	0.9	0.99	0.999	0.9999

x	-9	-99	-999	-9999
y	1.1	1.01	1.001	1.0001

You can see that the behaviour of $y = \dfrac{x-2}{x-1}$ needs to be described separately as $x \to +\infty$ and as $x \to -\infty$. In particular:

1. As x tends to $+\infty$ (becomes positive large without bound), y approaches 1 *from below* . This is written in symbols as[5]

$$x \to +\infty \Rightarrow y \to 1^-.$$

2. As x tends to $-\infty$ (becomes negative large without bound), y approaches 1 *from above* . This is written in symbols as[6]

$$x \to -\infty \Rightarrow y \to 1^+.$$

[5] $y \to 1^-$ means *y approaches 1 through values which are less than 1.*

[6] $y \to 1^+$ means *y approaches 1 through values which are greater than 1.*

The behaviour of a rational function as $x \to \pm\infty$ can be found directly from the function after a little rearrangement. As[7]

$$y = \frac{x-2}{x-1} = 1 - \frac{1}{x-1} \ ,$$

you can see that

1. when x is very large *and positive*, then

$$y = \frac{x-2}{x-1} = 1 - \frac{1}{x-1}$$

is very close to 1 *and below 1*, so $x \to +\infty \Rightarrow y \to 1^-$.

2. when x is very large *and negative*, then

$$y = \frac{x-2}{x-1} = 1 - \frac{1}{x-1}$$

is very close to 1 *and above 1*, so $x \to -\infty \Rightarrow y \to 1^+$.

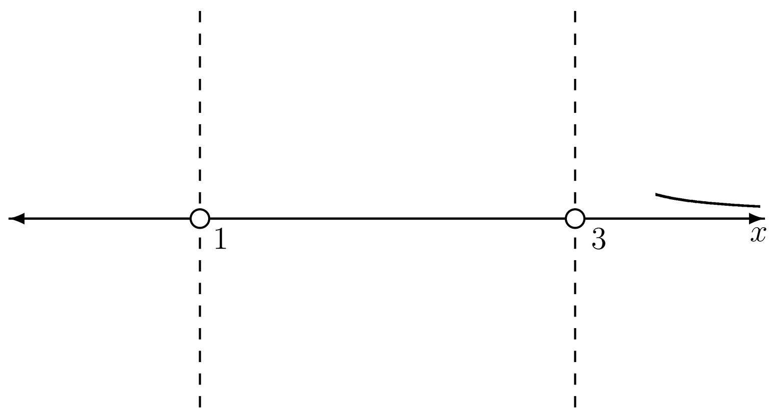
Example

$x \to \pm\infty$
$\Rightarrow y \to c$

What is the horizontal asymptote of $y = \dfrac{10}{(x-1)(x-3)}, x \neq 1, 3$? Sketch the graph near the horizontal asymptote.

Answer[8]

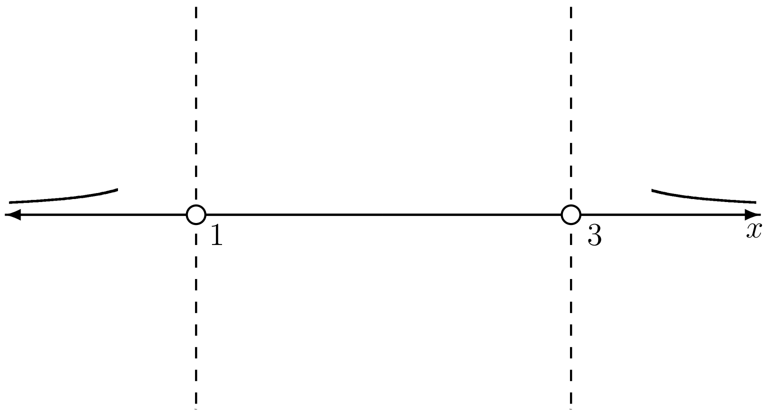
(1) $x \to +\infty \Rightarrow y \to 0^+$



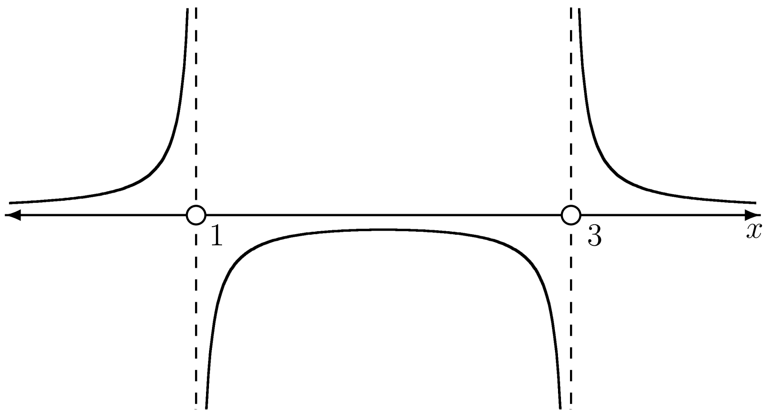
[7] See *Polynomials* (Module 1, section 1.3)

[8] Note that the curve does not cross the vertical asymptotes $x = 1$ and $x = 3$.

(2) $x \to -\infty \Rightarrow y \to 0^+$



The actual graph of $y = \dfrac{10}{(x-1)(x-3)}, x \neq 1, 3$ is ...



Exercise B.2

1. What are the horizontal asymptotes of

   (a) $y = 1 + \dfrac{1}{x-2}, x \neq 2$

   (b) $y = \dfrac{1}{x^2-1}, x \neq \pm 1$

   (c) $y = \dfrac{x}{x-1}, x \neq 1$

   (d) $y = 1 + \dfrac{x-1}{x+1}, x \neq \pm 1$

2. Sketch each of the graphs above near their horizontal asymptotes.

B.3 Other asymptotes

The graph of a rational functions can have an oblique asymptote (a non-horizontal straight line) or even be asymptotic to a simple curve. Examples are given below.

an oblique asymptote

Example

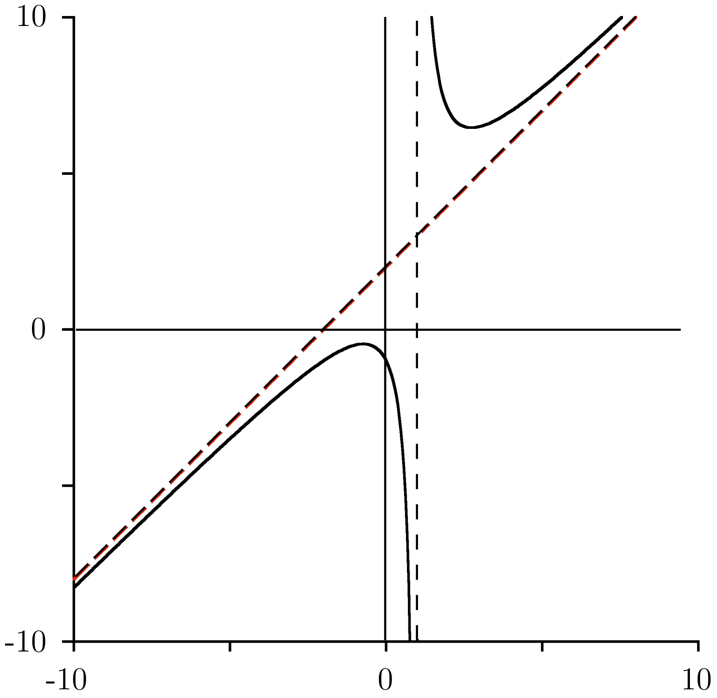
The graph of $y = \dfrac{x^2+x+1}{x-1}, x \neq 1$ has vertical asymptote $x = 1$.

Rewriting the function as [9]

$$y = \frac{x^2+x+1}{x-1} = x + 2 + \frac{3}{x-1}$$

shows that as $x \to \pm\infty$ the curve becomes very close to the line $y = x + 2$. This is written as

$$y \sim x + 2.$$ [10]



Asymptotes are important as they show how functions behave when the variable is given very large values. In the example above, we can see that $\dfrac{x^2+x+1}{x-1} \sim x + 2$ when $x \to \pm\infty$.

[9] See *Polynomials* (Module 1, section 1.3)

[10] The symbol $\sim$ means *is asymptotic to.*

Example

a curve can be an asymptote

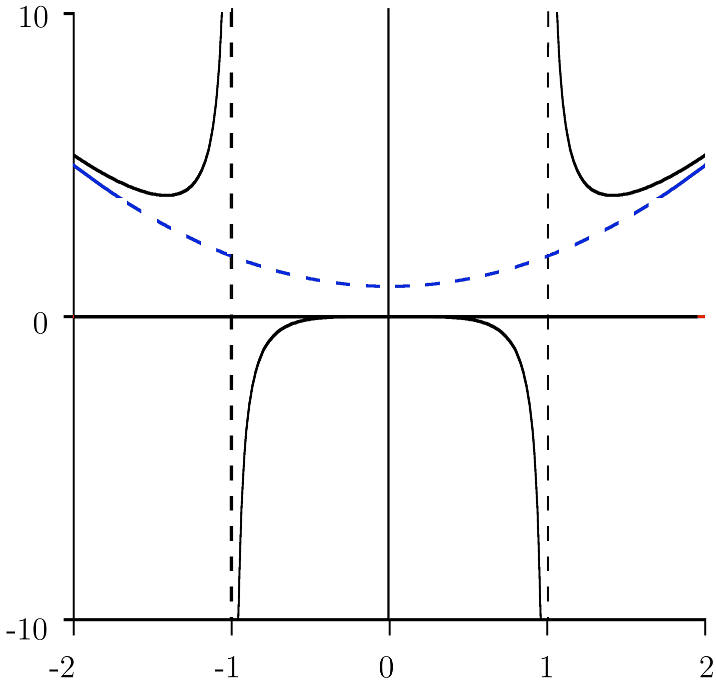
The graph of $y = \dfrac{x^4}{x^2-1}, x \neq \pm 1$ below has vertical asymptotes $x = 1$ and $x = -1$.

Rearranging as

$$y = \frac{x^4}{x^2-1} = x^2 + 1 + \frac{1}{x^2-1}$$

shows that the curve becomes very close to the parabola $y = x^2 + 1$ when $x \to \pm\infty$. This is written as

$$y \sim x^2 + 1.$$



Exercise B.3

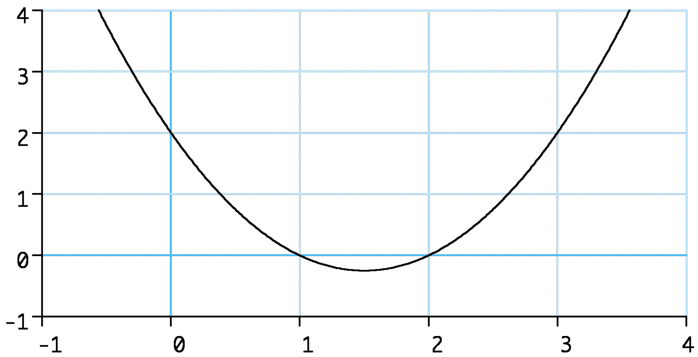
1. What are the oblique asymptotes of
   (a) $y = \dfrac{x^2}{x+1}, x \neq -1$
   (b) $y = \dfrac{x^3}{x^2+1}$
2. (a) What are the intercepts and asymptotes of $y = \dfrac{x^2-4}{x^2-1}, x \neq \pm 1$?
   (b) Use this information to draw a rough sketch of the graph. *Remember that a graph never cuts its asymptotes.*
3. Find a polynomial curve that approximates $y = \dfrac{x^4}{x^2+1}$ when $x \to \pm\infty$.

Appendix C

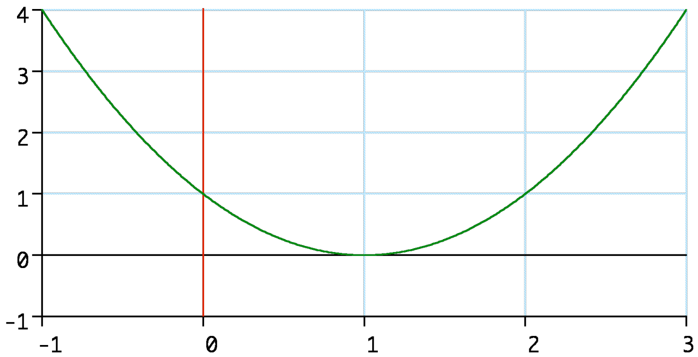
Answers

Exercise 1.1

1(a)

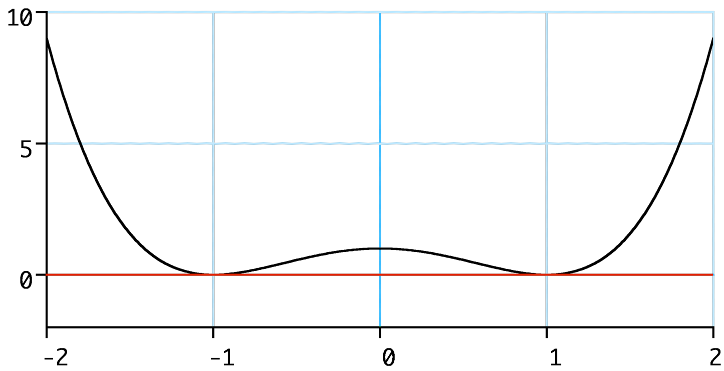


1(b)



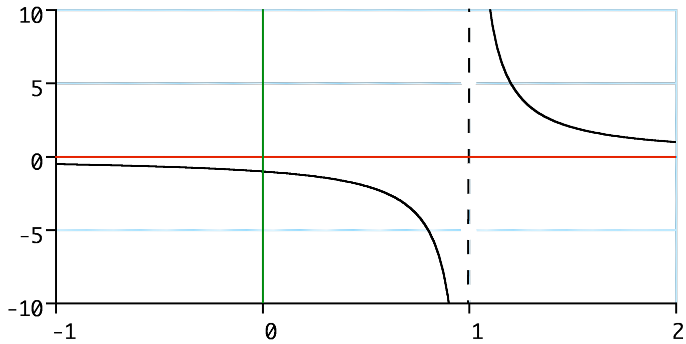
1(c)

1(d)

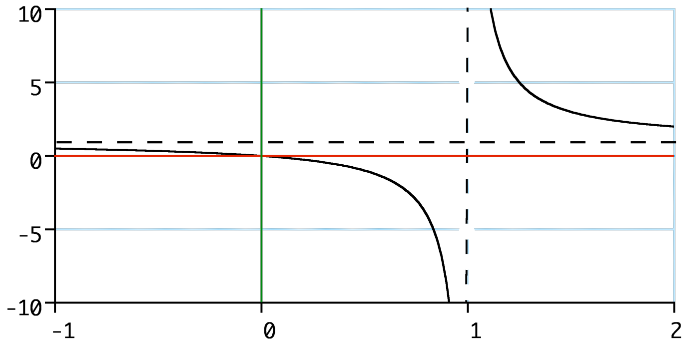


2. $y = x(x+1)^2(x-2)$

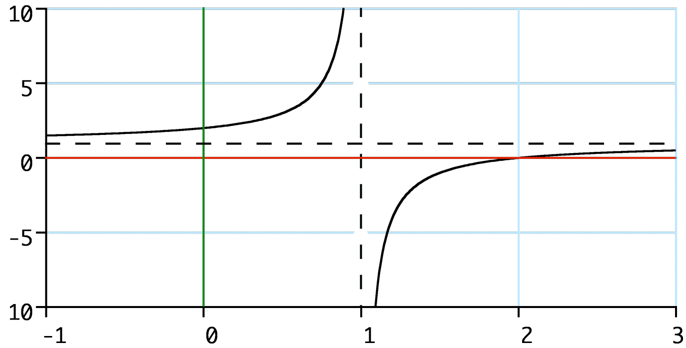
3(a) Vertical asymptote: $x = 1$; horizontal asymptote: $y = 0$.



3(b) Vertical asymptote: $x = 1$; horizontal asymptote: $y = 1$.

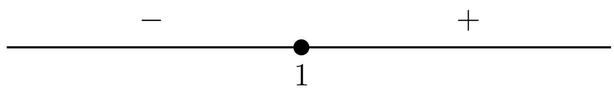


3(c) Vertical asymptote: $x = 1$; horizontal asymptote: $y = 1$.

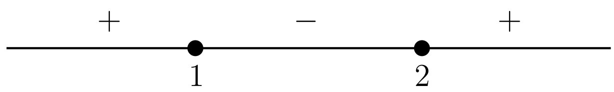


Exercise 1.2

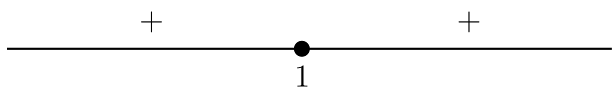
1(a)



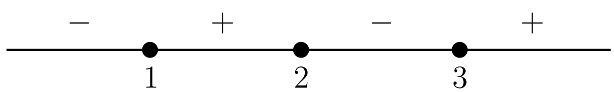
1(b)



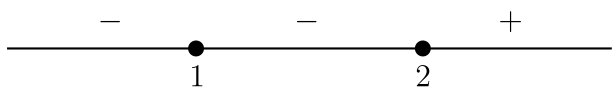
1(c)



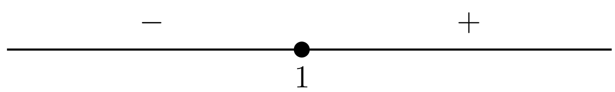
1(d)

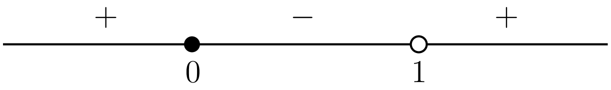


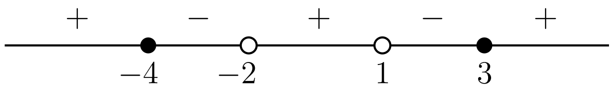
1(e)



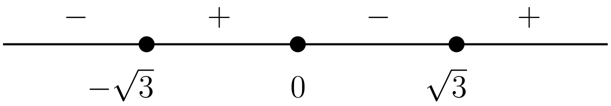
1(f)

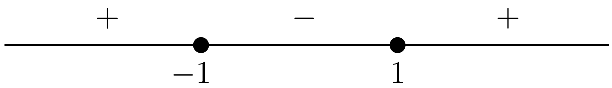


1(g) 

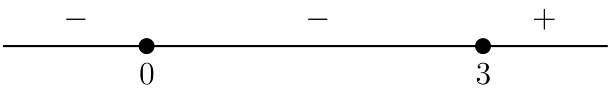
1(h) 

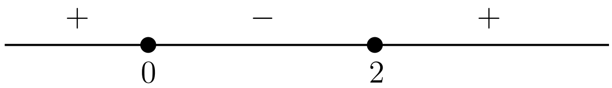
Exercise 1.3

1(a) 

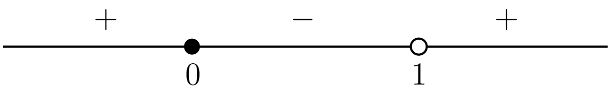
1(b) 

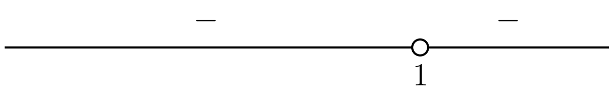
1(c) (i) $[-\sqrt{3}, 0] \cup [\sqrt{3}, \infty)$ (ii) $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$ (iii) $(-\infty, -1] \cup [1, \infty)$
(iv) $[-1, 1]$

2(a) 

2(b) 

2(c) (i) $\{0\} \cup [3, \infty)$ (ii) $(-\infty, 0) \cup (0, 3)$ (iii) $(-\infty, 0] \cup [2, \infty)$ (iv) $[0, 2]$

3(a) 

3(b) 

3(c) (i) $[-\infty, 0] \cup (1, \infty)$ (ii) $(0, 1)$ (iii) $\emptyset$ (nowhere)[1] (iv) $(-\infty, 1) \cup (1, \infty)$

Exercise 1.4

1(a) Turning points: $(\frac{2}{\sqrt{3}}, -\frac{16}{3\sqrt{3}})$, $(-\frac{2}{\sqrt{3}}, \frac{16}{3\sqrt{3}})$

[1] $\emptyset$ is the symbol for the empty set

1(b) Turning points: $(1, 11)$, $(5, -21)$

1(c) Stationary points: none

1(d) Turning points: $(-3, -81)$, $(0, 0)$, $(3, -81)$

1(e) Stationary points: none

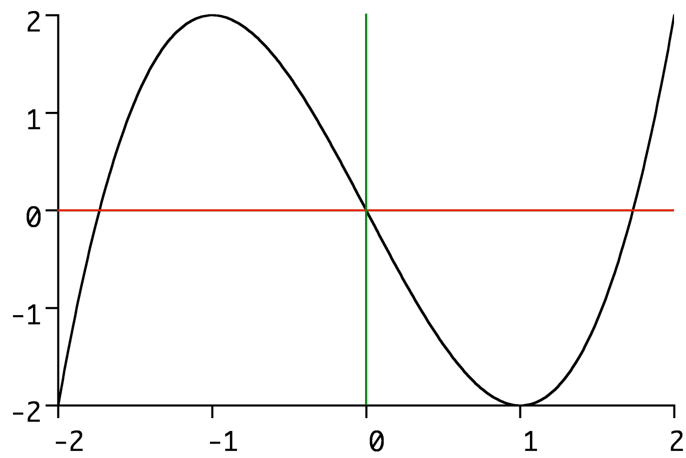
1(f) Turning points: $(0, -1)$, $(2, 3)$

2(a)	local max: $(0, -2/\sqrt{3})$	local min: $(0, 2/\sqrt{3})$
2(b)	local max: $(1, 11)$	local min: $(5, -21)$
2(d)	local max: $(0, 0)$	global min: $(-3, -81)$, $(3, -81)$
2(f)	local max: $(0, -1)$	local min: $(2, 3)$

3(a) $a = 3$

3(b) $(1, -2)$ and $(-1, 2)$

3(c)



4. $x = -\dfrac{b}{2a}$, $a < 0$; $a > 0$

5. $a = -3$, $b = 6$; $(-1, 8)$

6. $a = 1$, $b = 3$, $c = -9$ and $d = 5$

Exercise 1.5

1(a) point of inflection: $(0, 0)$ concave up: $[0, \infty)$ concave down: $(-\infty, 0])$
1(b) point of inflection: $(3, -5)$ concave up: $[3, \infty)$ concave down: $(-\infty, 3]$
1(c) point of inflection: $(0, 0)$ concave up: $(-\infty, 0]$ concave down: $[0, \infty)$
1(d) points of inflection: $\left(-\sqrt{\frac{3}{2}}, -\frac{45}{4}\right)$, $\left(\sqrt{\frac{3}{2}}, \frac{45}{4}\right)$
concave up: $\left(-\infty, -\sqrt{\frac{3}{2}}\right] \cup \left[\sqrt{\frac{3}{2}}, \infty\right)$ concave down: $\left[-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right]$
1(e) points of inflexion: none concave up: $(1, \infty)$ concave down: $(-\infty, 1)$
1(f) points of inflexion: none concave up: $(1, \infty)$ concave down: $(-\infty, 1)$

Exercise 2.2

1(a) $y = 200 - x$ 1(c) $A'(x) = 200 - 2x$; stationary point at $x = 100$. 1(d) $A(100) = 10,000\ \text{m}^2$

2(b) $L'(x) = 2 - \dfrac{200}{x^2}$ 2(c) $x = 10$ and $L(10) = 40$ m

3(c) $r = \left(\dfrac{500}{\pi}\right)^{1/3}$ and $A = 6\pi\left(\dfrac{500}{\pi}\right)^{2/3}$

4. width = length = $2\sqrt{2}$cm

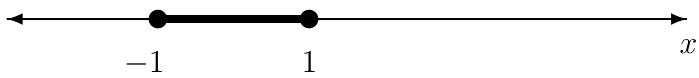
5. $(x, y) = (1, 2)$ and $A = 4$

6(b) $V = \dfrac{1}{3}\pi\left(\dfrac{\theta}{36}\right)^2\sqrt{100 - \left(\dfrac{\theta}{36}\right)^2}$

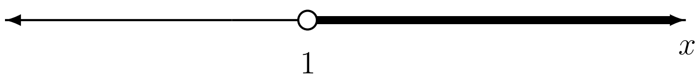
6(c) $\theta = 36\sqrt{\dfrac{200}{3}} \approx 293.9^\circ$ and $V = \dfrac{2000\pi}{9\sqrt{3}} \approx 403.1\ \text{cm}^2$

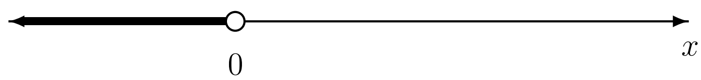
Exercise A

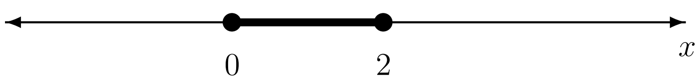
1(a) $(0, 10)$ or $\{x : 0 < x < 10\}$ 1(b) $(0, 10]$ or $\{y : 0 < x \le 10\}$
1(c) $[5, \infty)$ or $\{L : L \ge 5\}$ 1(d) $(-\infty, 0] \cup [5, 10)$ or $\{t : t \le 0\} \cup \{t : 5 \le t < 10\}$

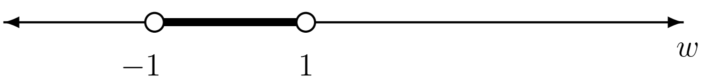
2(a)


2(b)


2(c)


2(d)


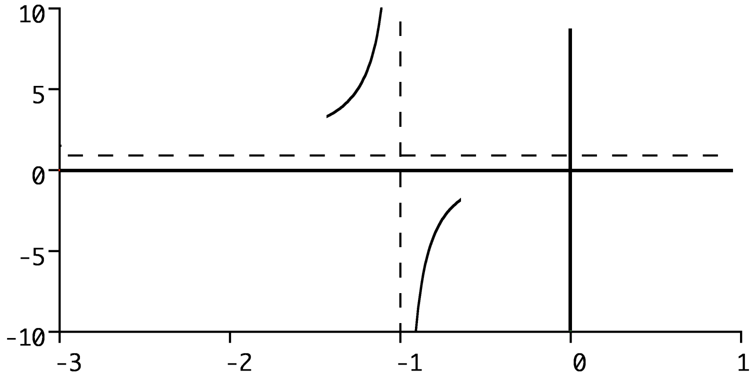
2(e)


2(f)


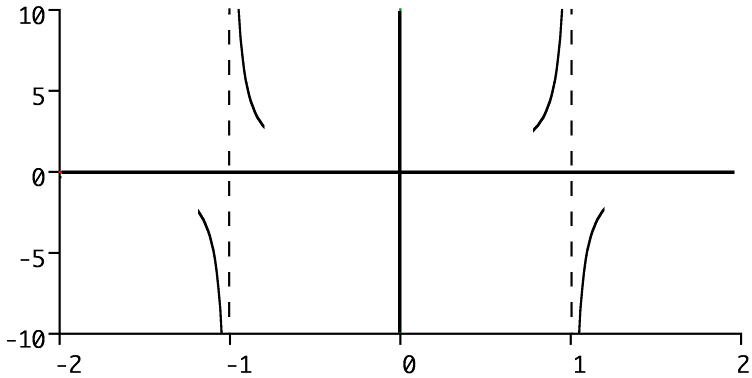
Exercise B.1

1(a) $x = -1$ 1(b) $x = 1$ and $x = -1$ 1(c) $x = 2$ 1(d) $x = 1$ and $x = -1$

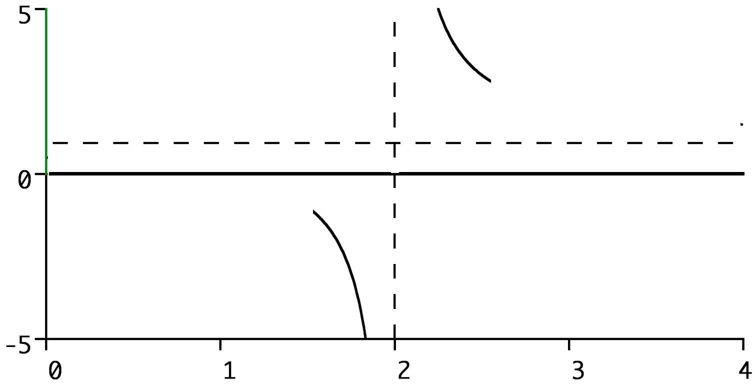
2(a)



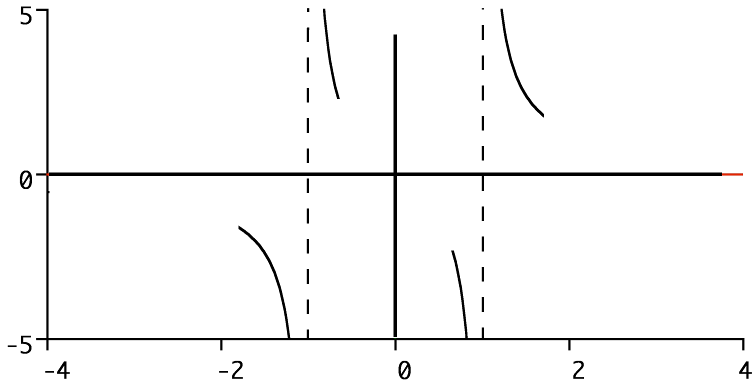
2(b)



2(c)



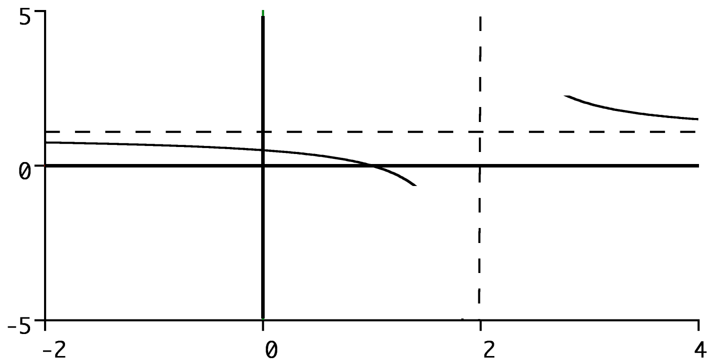
2(d)



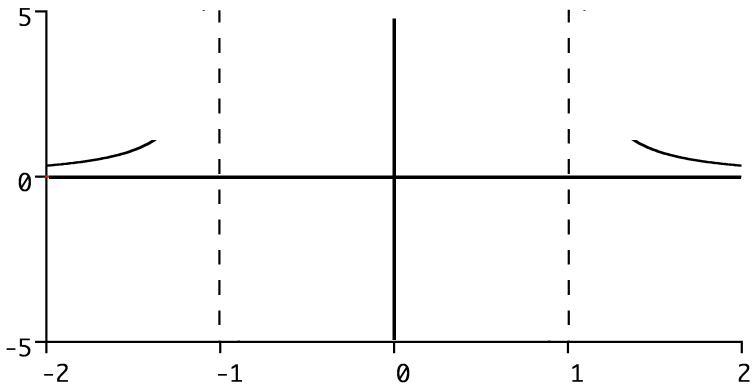
Exercise B.2

1(a) $x = 1$ 1(b) $x = 0$ 1(c) $x = 1$ 1(d) $x = 2$

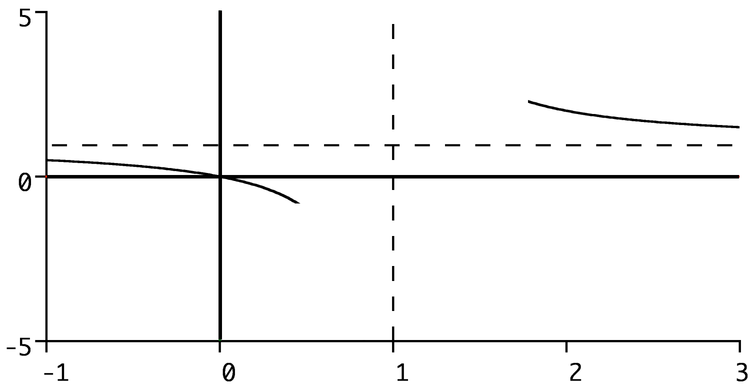
2(a)



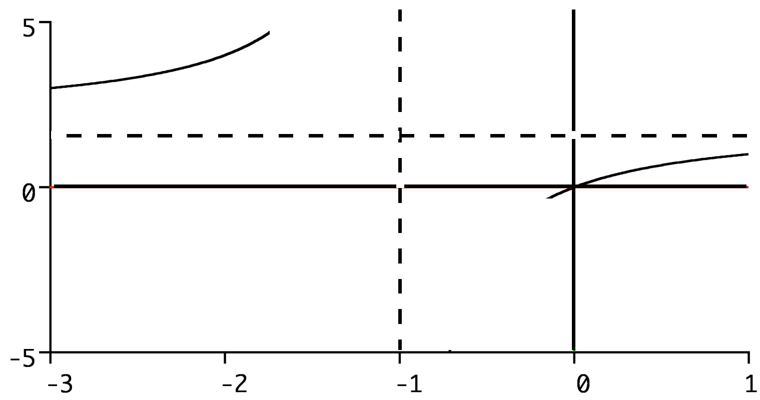
2(b)



2(c)



2(d)

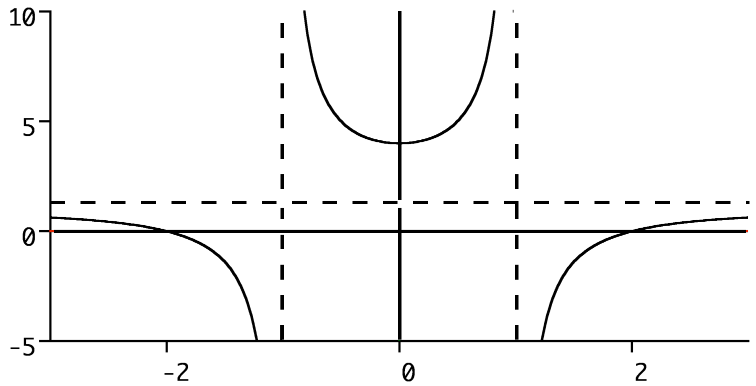


Exercise B.3

1(a) $y = x - 1$ 1(b) $y = x$

2(a) $(\pm 2, 0)$, $(0, 4)$; $y = x$

2(b)



3. As

$$y = \frac{x^4}{x^2+1} = x^2 - 1 + \frac{1}{x^2+1} \ ,$$

the curve is $y = x^2 - 1$.