Useful series for probability proofs

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Sum of natural numbers

$$\sum_{n=1}^{N} n = \frac{1}{2}N(N+1) = \frac{1}{2}N^2 + \frac{1}{2}N$$

Sum of squares of natural numbers

$$\sum_{n=1}^{N} n^2 = \frac{1}{6}N(N+1)(2N+1) = \frac{1}{3}N^3 + \frac{1}{2}N^2 + \frac{1}{6}N$$

Finite geometric series

$$\sum_{n=0}^{N} x^n = \frac{x^{N+1}-1}{x-1} = \frac{1-x^{N+1}}{1-x}$$

Infinite geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{(as long as } |x|<1\text{)}$$

Maclaurin series for e^x

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

Binomial expansion

$$\sum_{n=0}^{N} \binom{N}{n} x^n y^{N-n} = (x+y)^N$$

Note: $\binom{N}{n} = \dfrac{N\cdot(N-1)\cdot\ldots\cdot\big(N-(n-2)\big)\cdot\big(N-(n-1)\big)}{n\cdot(n-1)\cdot\ldots\cdot 2\cdot 1}$ (n factors on both top and bottom)

$$= \frac{N!}{n!\,(N-n)!}$$

Infinite binomial series

$$\sum_{n=0}^{\infty} \binom{N}{n} x^n = (1+x)^N$$

Note: $\binom{N}{n} = \dfrac{N\cdot(N-1)\cdot\ldots\cdot\big(N-(n-2)\big)\cdot\big(N-(n-1)\big)}{n\cdot(n-1)\cdot\ldots\cdot 2\cdot 1}$ (n factors on both top and bottom)

with $\binom{N}{0} = 1$

Note: This applies when N is not a natural number too, but only if $|x| < 1$.